**2009, Spring**

**Incomplete:** 3, 4.

**Problem 1.**

(a) We consider the cases of finite and infinite countable unions separately.

- Suppose $\{E_j\}_{j=1}^m \subset \mathcal{C}$ and let $\epsilon > 0$. For each $1 \le j \le m$, there's a set $A_j \in \mathcal{A}$ such that $A_j \subset E_j$ and $\mu(E_j \setminus A_j) < \epsilon/m$. Note that $A := \bigcup_{j=1}^m A_j \in \mathcal{A}$ since $\mathcal{A}$ is an algebra, and we have $A \subset E := \bigcup_{j=1}^m E_j$. Then

$$\mu(E \setminus A) = \mu\Big(\bigcup_{j=1}^m (E_j \setminus A)\Big) \le \mu\Big(\bigcup_{j=1}^m (E_j \setminus A_j)\Big) \le \sum_{j=1}^m \mu(E_j \setminus A_j) < \sum_{j=1}^m \frac{\epsilon}{m} = \epsilon,$$

so $E \in \mathcal{C}$.

- Now suppose $\{E_j\}_{j=1}^\infty \subset \mathcal{C}$ and let $\epsilon > 0$. Letting $F_m := \bigcup_{j=1}^m E_j$ for each $m \in \mathbb{N}$, we have an increasing sequence $F_1 \subset F_2 \subset \cdots$ with $F_m \nearrow E := \bigcup_{j=1}^\infty E_j$ as $m \to \infty$, so by continuity from below, $\mu(F_m) \to \mu(E)$ as $m \to \infty$. Because $\mu(E) \le \mu(X) < \infty$, we can choose $m \in \mathbb{N}$ large enough so that $\mu(E) - \mu(F_m) < \epsilon/2$, whereby

$$\mu(E) = \mu(E \setminus F_m) + \mu(F_m) \implies \mu(E \setminus F_m) = \mu(E) - \mu(F_m) < \epsilon/2,$$

the first equality holding since $F_m \subset E$. Moreover, $F_m \in \mathcal{C}$ by the above argument, so we can find some $A \in \mathcal{A}$ with $A \subset F_m \subset E$ and $\mu(F_m \setminus A) < \epsilon/2$. Then

$$\mu(E) = \mu(E \setminus F_m) + \mu(F_m) \implies \mu(E \setminus A) \le \mu(E \setminus F_m) + \mu(F_m \setminus A) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

and thus $E \in \mathcal{C}$.

□

(b) Let $X := [0, 1]$ with $\sigma$-algebra $\mathcal{B}_{[0,1]}$ and Lebesgue measure $\mu$. Let $\mathcal{A} \subset \mathcal{B}_{[0,1]}$ be the algebra generated by all singletons $\{q\}$, $q \in E := \mathbb{Q} \cap [0, 1]$, using complements and finite unions. Then $A \in \mathcal{A}$ if and only if $A$ is a finite collection $\{q_j\}_{j=1}^m \subset E$ or $A$ is the complement of such a set. Note that $\{0\} \in \mathcal{A}, \{0\} \subset E$, and $\mu(E \setminus \{0\}) \le \mu(E) = 0 < \epsilon$ for any $\epsilon > 0$, so $E$ is approximable from inside by $\mathcal{A}$. But observe that any element $A \in \mathcal{A}$ contains at least one rational, while $E$ contains only irrationals, so we can't have $A \subset E^{\mathsf{c}}$, and thus $E^{\mathsf{c}}$ isn't approximable from inside by $\mathcal{A}$.

**Problem 2.**

(a) Both $f, g$ are continuous on the compact set $[a, b]$, so there's some $M > 0$ large enough so that $|f|, |g| \le M$ on all of $[a, b]$. Now let $\epsilon > 0$ and choose $\delta > 0$ such that for any disjoint collection $\{(a_j, b_j) \subset [a, b]\}_{j=1}^N$, we have

$$\sum_{j=1}^N (b_j - a_j) < \delta \implies \sum_{j=1}^N |f(b_j) - f(a_j)|, \sum_{j=1}^N |g(b_j) - g(a_j)| < \frac{\epsilon}{2M}.$$

Then for any such collection,

$$\sum_{j=1}^{N}|f(b_j)g(b_j)-f(a_j)g(a_j)| \leq \sum_{j=1}^{N}\left[|f(b_j)g(b_j)-f(b_j)g(a_j)|+|f(b_j)g(a_j)-f(a_j)g(a_j)|\right]$$

$$\leq M\Bigg(\underbrace{\sum_{j=1}^{N}|g(b_j)-g(a_j)|}_{<\epsilon/2M}+\underbrace{\sum_{j=1}^{N}|f(b_j)-f(a_j)|}_{<\epsilon/2M}\Bigg) < M\left(\frac{\epsilon}{2M}+\frac{\epsilon}{2M}\right)=\epsilon.$$

□

(b) We've just seen that $fg$ is absolutely continuous, so we have

$$f(b)g(b)-f(a)g(a)=\int_a^b (fg)'=\int_a^b (f'g+fg')=\int_a^b f'g+\int_a^b fg'$$

by the fundamental theorem for Lebesgue integrals. □

(c) Take some $[a,b]\subset\mathbb{R}$ with $b-a\neq 2$, and let $f,g:[a,b]\to\mathbb{R}$ be given by $f(x):=(x-a)/(b-a)$ and $g(x):=\frac{1}{2}\mathbb{1}_{[\frac{b-a}{2},b]}(x)$. Then $f'=1$ and $g'=0$ a.e. on $[a,b]$, but $g$ isn't continuous (in particular, $g$ isn't absolutely continuous). We have

$$\int_a^b \underbrace{f'}_{=1} g+\int_a^b f\underbrace{g'}_{=0}=\int_a^b g=\frac{b-a}{4}\neq\frac{1}{2}=\underbrace{f(b)}_{=1}\underbrace{g(b)}_{=1/2}-\underbrace{f(a)g(a)}_{=0}.$$

□