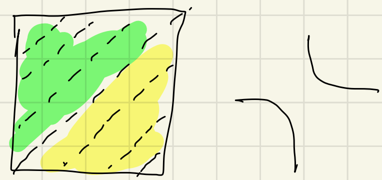
08 sp 1

$E = [0,1] \times [0,1]$



(i) $\displaystyle\int_E \frac{1}{x-y}\,dm(x,y)$

Def $f = \dfrac{1}{x-y}$,

$$\int_E \frac{1}{x-y}\,dm(x,y) = \int_{E,\, x>y} f - \int_{E,\, y>x} -f$$

$$= \infty - \infty$$

$\Rightarrow$ integral does not exist. $\square$

Folland requires at least one of $\int f^+$, $\int f^-$ to be finite.

(ii) $\displaystyle\int_E \frac{1}{x+y}\,dm(x,y)$

Def $f = \dfrac{1}{x+y}$. $f \in$ measurable $L^+(E)$

$$\Rightarrow \int_E f\,dm(x,y) = \int_0^1\int_0^1 f\,dx\,dy$$

$$= \int_0^1 \Big[\ln(x+y)\Big]_0^1 dy$$

$$= \int_0^1 \ln(y+1) - \ln(y)\; dy$$

$u = \ln(y+1)$. $dv = dy$

$du = \dfrac{1}{y+1}\,dy \quad v = y+1$

$$= \Big[(y+1)\ln(y+1) - y\Big]_0^1 - \Big[y\ln(y) - y\Big]_0^1$$

$$= 2\ln(2) - \cancel{1} - \ln(1) + \cancel{0} - \Big[\ln(1) - \cancel{1} - \cancel{0\ln(0)} \cancel{+0}\Big]$$

$\lim_{y\to 0} y\ln(y) = 0 \cdot -\infty$ gives 0

$$= 2(\ln 2 - \ln 1) = 2\ln(2) = \ln(4). \quad \square$$

08 sp. 2

2. Let f be a nonnegative measurable function on [0,1] such that $\int_0^1 f(x)\,dx = 1$. Define a measure μ on $[0,1]$ by
$$\mu(A) = \int_A f(x)\,dx, \quad A \in \mathcal{B}([0,1]).$$
Let K be the intersection of all compact subsets E of $[0,1]$ such that $\mu(E) = 1$. Find $\mu(K)$.

Folland 4.21: X compact $\iff$ for every family $\{F_\alpha\}_{\alpha \in A}$ of closed sets with the finite intersection property, $\bigcap_{\alpha\in A} F_\alpha \neq \emptyset$.
A family of subsets of X is said to have the finite intersection property if $\bigcap_{\alpha \in B} F_\alpha \neq \emptyset$ $\forall$ finite $B \subset A$.

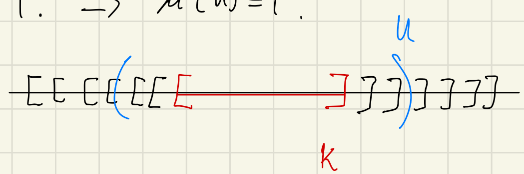
To see this, look at $[0,1]$ and $(0,1)$.
Family $\{[0, 1/n]\}$ has finite intersection prop. and non empty intersection.
Family $\{(0, 1/n]\}$ in $(0,1)$ has finite intersection prop. but empty intersection.
$[0,1]$ compact, $(0,1)$ not compact.

$U \supset K$, open. U^c closed $\Rightarrow$ U^c compact.
$\Rightarrow$ any open cover of U^c has a finite subcover.
$U \supset K \Rightarrow U^c \subset K^c = (\bigcap E)^c = \bigcup E^c$, an open cover of U^c.
$\Rightarrow \exists$ finite subcover $\{E_i^c\}_1^m$ of U^c.
$\bigcup_1^m E_i^c \supset U^c \Rightarrow (\bigcup_1^m E_i^c)^c \subset U \Rightarrow \bigcap_1^m E_i \subset U$.

Now, let E_1, E_2 be compact with $\mu(E_1) = \mu(E_2) = 1$.
Then $\mu(E_1 \cap E_2) = \int_{E_1\cap E_2} f = \int_{X - (E_1^c \cup E_2^c)} f = \int_X f - \int_{E_1^c} f - \int_{E_2^c} f$

$$= 1 - \mu(E_1^c) - \mu(E_2^c) = 1 - 0 - 0 = 1.$$

Induction shows then $\mu(\bigcap_1^m E_i) = 1$. $\Rightarrow \mu(U) = 1$.
$\mu(K) = \inf\{\mu(U) : U \text{ open}, U \supset K\}$
since μ is a Lebesgue-Stieltjes measure. $\Rightarrow \mu(K) = 1$.

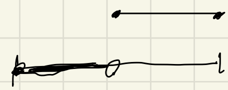


$\square$

08sp. 3

$f : [0,1] \to \mathbb{R}$ $\qquad$ f cont. a.e. $\overset{?}{\Longleftrightarrow}$ $\exists$ cont. g s.t. $f = g$ a.e.

$\not\Longrightarrow$ $\quad$ Let $\quad f = \begin{cases} 0 & x \in [0, 1/2) \\ 1 & x \in [1/2, 1] \end{cases}$



Let $g : [0,1] \to \mathbb{R}$ be s.t. $f = g$ a.e.
Let $x \in [0, 1/2)$. Assume $g(x) \neq 0$.
g continuous $\Rightarrow$ $\forall\ \varepsilon > 0$, $\exists\ \delta > 0$ s.t. $y \in (x-\delta, x+\delta) \cap [0,1]$
$\Rightarrow$ $g(y) \in (g(x) - \varepsilon, g(x) + \varepsilon)$. Take $\varepsilon = |g(x)|/2$.
Then $g(y) \neq 0$ $\forall$ $y \in (x-\delta, x+\delta)$ $\Rightarrow$ $f \neq g$ a.e. $\Rightarrow\Leftarrow$.
So $g(x) = 0$ $\forall$ $x \in [0, 1/2)$.
Similarly, $g(x) = 1$ $\forall$ $x \in [1/2, 1]$.
That is $g = f$, so g not cont.

$\not\Longleftarrow$ $\quad$ Let $g = 1$. Let $f = \begin{cases} 1 & x \notin \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$

$m([0,1] \cap \mathbb{Q}) = 0 \Rightarrow f = g$ a.e.
But f is continuous nowhere.

Indeed, take $\varepsilon = 1/2$. There is no $\delta > 0$
s.t. $\quad y \in (x-\delta, x+\delta) \Rightarrow f(y) \in (1/2, 3/2)$
or $f(y) \in (-1/2, 1/2)$.

$\square$

08 sp. 4

$f \in L^1(\mathbb{R})$. Claim:

$$\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)dx \right| = \|f\|_{L^1(\mathbb{R})}.$$

$\|f\|_{L^1(\mathbb{R})} = \int_{-\infty}^{\infty} |f|$, call this value C.

e.g. $m=7$ $\quad \left|\int_{-7}^{-48/7} f(x)dx\right| + \left|\int_{-48/7}^{-47/7} f dx\right| + \dots + \left|\int_{7}^{50/7} f dx\right|.$

$$\text{LHS} \leq \lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \int_{k/m}^{(k+1)/m} |f(x)|dx$$
$$= \lim_{m\to\infty} \int_{-m}^{(m^2+1)/m} |f(x)|dx = \int_{-\infty}^{\infty} |f| dx = \|f\|_{L^1(\mathbb{R})}.$$

Let $\varepsilon > 0$. Is LHS $>$ RHS $- \varepsilon$?

First, assume $g \in L^1(\mathbb{R})$ cont. and vanishes outside bounded interval.

Define $g_m := \sum_{k=-m^2}^{m^2} m \left| \int_{k/m}^{(k+1)/m} g(x)dx \right| \chi_{[k/m, (k+1)/m)}$

we see $\lim_m \int_{\mathbb{R}} g_m(x)\,dx = \text{LHS}$. (with f replaced by g).

Claim: $\lim_m g_m = |g|$.

Let $x \in \mathbb{R}$. $g_m(x) = m \left| \int_{k/m}^{(k+1)/m} g(y)dy \right|$ s.t. $x \in [k/m, (k+1)/m)$.

Assume $g(x) \neq 0$ $\quad g$ cont. $\Rightarrow \exists M$ large enough s.t. $g(y)$ is the same sign as $g(x)$ for all $y \in [k/m, (k+1)/m)$, and all $m > M$.

$\Rightarrow g_m(x) = m \int_{k/m}^{(k+1)/m} |g(y)|dy \quad \forall m > M$.

By IVT, for each m here, $\exists y_m \in [k/m, (k+1)/m)$ s.t.

$|g(y_m)| = \frac{1}{m}\int_{k/m}^{(k+1)/m} |g(y)|\,dy$. These $y_m \to x$ as $m \to \infty$.

Then $\lim g_m(x) = \lim |g(y_m)| = |g(x)|$ since g cont.

all $g_m \leq \sup |g| \, \chi_{\{x: g(x) \neq 0\}} \in L^1(\mathbb{R})$. DCT $\Rightarrow$ result.

Since we have lebesgue measure on $\mathbb{R}$, we can always find such a g s.t. $\int |f-g| < \varepsilon \Rightarrow \int |g| > \int |f| - \varepsilon$.

We know $\exists$ m s.t. $\sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x)\,dx \right| > \int |g| - \varepsilon$.

$$\sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x) \right| - \left| \int_{k/m}^{(k+1)/m} f(x) \right|$$
$$\leq \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x) - f(x)\,dx \right|.$$
$$\leq \sum_{k=-m^2}^{m^2} \int_{k/m}^{(k+1)/m} |g-f|\,dx < \varepsilon.$$
$$\Rightarrow \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| > \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} g(x)\,dx \right| - \varepsilon$$

$$\Rightarrow \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| > \int |f| - 3\varepsilon.$$

Take $\varepsilon \to 0$, then

$$\lim_m \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| \geq \int |f|$$

$$\Rightarrow \lim_m \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{(k+1)/m} f(x)\,dx \right| = \int |f|. \qquad \square$$