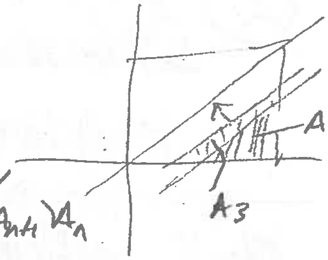
Spring '08:

(1.) m = Leb. meas on $E=[0,1]\times[0,1]$. Determine existence of the integrals: (i) $\int_E \frac{1}{x-y}\,dm$, (ii) $\int_E \frac{1}{x+y}\,dm$

(i) Consider the subsets $A_n=\{y\le x-\frac{1}{n}\}\subseteq E$, $n\geq 1$

Note that $A_n\subseteq A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = E\cap\{y\le x\}$.



Therefore: $\chi_{\bigcup_{n=1}^{\infty} A_n} = \chi_{\bigcup_{n=1}^{\infty} A_n\setminus A_{n+1}} = \sum_{n=1}^{\infty}\chi_{A_{n+1}\setminus A_n} = \lim_{k\to\infty}\sum_{n=1}^{k}\chi_{A_{n+1}\setminus A_n}$

$= \lim_{k\to\infty}\left(\sum_{n=1}^{k}(\chi_{A_{n+1}}-\chi_{A_n})\right) = \lim_{k\to\infty}\left(\chi_{A_2}-\chi_{A_1}+\chi_{A_3}-\chi_{A_2}+\dots+\chi_{k+1}-\chi_{A_k}\right)$

$= \lim_{k\to\infty}\left(\chi_{k+1}-\chi_{A_1}\right)$

Now: $\int_{E\cap\{y\le x\}}\frac{dm}{x-y} = \int_{\bigcup_{n=1}^{\infty}A_n}\frac{dm}{x-y} = \int_E \chi_{\bigcup_{n=1}^{\infty}A_n}\frac{dm}{x-y} = \int_E\left(\lim_{k\to\infty}\chi_{A_{k+1}}-\chi_{A_1}\right)\frac{dm}{x-y}$

$= \underline{\int_E \lim_{k\to\infty}\chi_{A_{k+1}}\frac{dm}{x-y}} - \cancel{\int_E \chi_{A_1}\frac{1}{x-y}\,dm} \to 0$ since $m(A_1)=0$

And since $\frac{1}{x-y}$ is positive on all the A_n's and $\frac{1}{x-y}\le n$ on each A_n, the sequence $\chi_{A_n}\frac{1}{x-y}$ is increasing, so we apply MCT:

$\underline{\int_E \lim_{k\to\infty}\chi_{k+1}\frac{dm}{x-y}} = \lim_{k\to\infty}\int\chi_{A_{k+1}}\frac{dm}{x-y} = \lim_{k\to\infty}\int_{A_{k+1}}\frac{dm}{x-y}$

But now we see that:

$\int_{A_n}\frac{dm}{x-y} = \int_{1/n}^{1}\int_0^{x-\frac{1}{n}}\frac{dy\,dx}{x-y} = -\int_{1/n}^{1}\ln|x-y|\Big]_0^{x-\frac{1}{n}}dx = -\int_{1/n}^{1}\left(\ln|\tfrac{1}{n}|-\ln|x|\right)dx$

$= -\int_{1/n}^{1}\ln|\tfrac{1}{nx}|\,dx = \int_{1/n}^{1}\ln|nx|\,dx = x\ln|nx|\Big]_{1/n}^{1} - \int_{1/n}^{1}x\left(\frac{n\,dx}{nx}\right) = \Big[x\ln|nx|-x\Big]_{1/n}^{1}$

$= \left[(1)\ln|n|-1\right]-\left[\tfrac{1}{n}\ln|1|-\tfrac{1}{n}\right] = \ln|n|+\tfrac{1}{n}-1 \longrightarrow \infty$ as $n\to\infty$

Hence $\int_{E\cap\{y\le x\}}\frac{dm}{x-y}=\infty$, hence $\int_E\frac{dm}{x-y}=\infty$.

(ii) By symmetry of $\frac{1}{x+y}$ on E, we have:

$$\int_E \frac{1}{x+y}\,dm = 2\int_{E\cap\{y\le x\}} \frac{1}{x+y}\,dm = 2\int_0^1\int_0^x \frac{dy\,dx}{x+y} = 2\int_0^1 (\ln|2x| - \ln|x|)\,dx$$

$$= 2\int_0^1 \ln\left|\frac{2x}{x}\right| dx = 2\int_0^1 \ln|2|\,dx = \underline{2\ln|2|}$$

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(2) $f \geq 0$ measurable on $[0,1]$ s.t. $\int_0^1 f(x)\,dx = 1$

Define the measure μ on $[0,1]$ by $\mu(A) = \int_A f(x)\,dx$, $A \in \mathcal{B}[0,1]$

Let $K = \bigcap\{E \subseteq [0,1] \text{ cpt}, \mu(E) = 1\}$; $\mu(K) = ?$

<u>Step 1: $\mu(E_1 \cap E_2) = 1$ for E_1, E_2 cpt, meas 1</u>

Let $E_1, E_2 \subseteq [0,1]$ cpt, $\mu(E_1) = \mu(E_2) = 1$.

Now, $E_1 \cap E_2$, $E_1 \cup E_2 \subseteq [0,1]$, hence $\mu(E_1 \cap E_2) \le 1$, $\mu(E_1 \cup E_2) \le 1$, and thus: $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cap E_2)$

$$\Longrightarrow \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2)$$

$$= 1 + 1 - \mu(E_1 \cup E_2) \geq 2 - 1 = 1$$

Hence $\mu(E_1 \cap E_2) = 1$

<u>Step 2: $K = \bigcap_\alpha E_\alpha$ where E_α cpt, $\mu(E_\alpha) = 1$ has meas 1</u>

Recall that $\mu(K) = \inf\{\mu(U) : K \subseteq U,\ U \text{ open}\}$. $(*)$

Let $K = \bigcap_\alpha E_\alpha \subseteq U$ and let $F_\alpha = E_\alpha \cap U^c$. Since $\bigcap_\alpha E_\alpha \subseteq U$, we have that $\bigcap_\alpha F_\alpha = \emptyset$.

Note that F_α is intersection of two closed sets, hence closed, hence $\{F_\alpha\}$ is a family of closed sets in $[0,1]$ with $\bigcap_\alpha F_\alpha = \emptyset$;

$[0,1]$ is compact, hence every family of closed sets with the finite-intersection prop. has a nonempty intersection; hence $\{F_\alpha\}$ does <u>not</u> have the finite intersection prop., i.e. $\exists$ finite subfamily $\{F_{\alpha_1}, \dots, F_{\alpha_n}\}$ s.t. $\bigcap_{i=1}^n F_{\alpha_i} = \emptyset$. Hence $\exists$ finite collection $\{E_{\alpha_1}, \dots, E_{\alpha_n}\}$ s.t. $\bigcap_{i=1}^n E_{\alpha_i} \subseteq U$, hence $\mu\left(\bigcap_{i=1}^n E_{\alpha_i}\right) \le \mu(U)$. By Part 1, a finite int. of cpt, meas 1 sets is meas 1, hence $\mu(U) \geq 1$, hence $\underline{\mu(K) = 1}$ by $(*)$.

(3.) $f:[0,1]\to\mathbb{R}$.

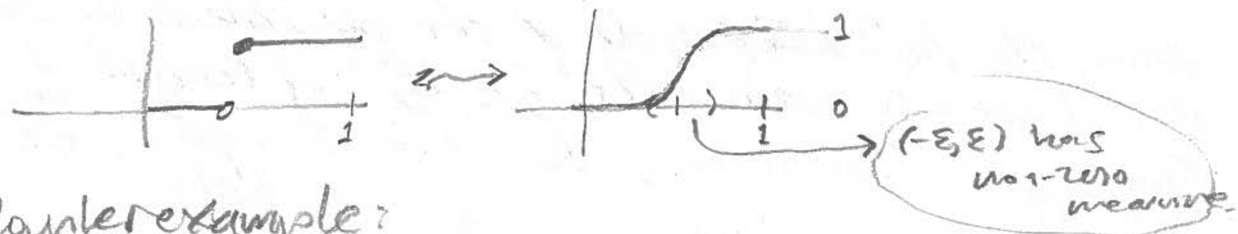
I. f continuous a.e. on $[0,1]$

II. $\exists$ continuous $g:[0,1]\to\mathbb{R}$ s.t. $f=g$ a.e.

Determine if I $\Rightarrow$ II or II $\Rightarrow$ I.

I $\not\Rightarrow$ II : Counterexample: let $f(x)=\chi_{[\frac{1}{2},1]}(x)$

- set of discontinuities of $f=\{\frac{1}{2}\}$, hence set is measure 0, hence f is cont. a.e.
- Any continuous function approximating f will have to have smoothing near the jump discontinuity w/ non-zero measure:



II $\not\Rightarrow$ I : Counterexample:

$$f(x)=\begin{cases}0 & x\in\mathbb{Q}\\ 1 & x\in\mathbb{R}\setminus\mathbb{Q}\end{cases}\quad;\quad g(x)\equiv 1$$

$f=g$ a.e., but f is nowhere continuous.

4. $f \in L^1(\mathbb{R})$. Prove that $\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{\frac{(k+1)}{m}} f(x)\,dx \right| = \|f\|_{L^1}$

Since $f \in L^1(\mathbb{R})$ with Lebesgue measure, $\exists$ continuous $g \in L^1$ such that $\|f-g\|_{L^1} = \int_{\mathbb{R}} |f-g|\,dx < \varepsilon$ for each $\varepsilon > 0$.

Hence: for $\varepsilon > 0$, $\exists$ continuous g s.t.

(1) $\varepsilon > \int_{\mathbb{R}} |f-g| > \int_{k/m}^{k+1/m} |f-g| > \int_{k/m}^{k+1/m} |f| - \int_{k/m}^{k+1/m} |g|$, and

(2) $\varepsilon > \int_{\mathbb{R}} |g-f| > \int_{k/m}^{k+1/m} |g-f| > \left| \int_{k/m}^{k+1/m} g-f \right| = \left| \int_{k/m}^{k+1/m} g - \int_{k/m}^{k+1/m} f \right|$

$$> \left| \int_{k/m}^{k+1/m} g \right| - \left| \int_{k/m}^{k+1/m} f \right|$$

Now, the left hand side of the desired equality is a sum over intervals $\left[\frac{k}{m}, \frac{k+1}{m}\right]$ of length $\frac{1}{m}$. Since g is continuous, for large enough m we have that $g \geq 0$ or $g \leq 0$ on each $\left[\frac{k}{m}, \frac{k+1}{m}\right]$.

$\to$ Hence for $m \to \infty$, $\int_{k/m}^{k+1/m} |g| = \left| \int_{k/m}^{k+1/m} g \right|$

See that (1) $\int_{k/m}^{k+1/m} |f| - \int_{k/m}^{k+1/m} |g| < \varepsilon$

for large m

(2) $\int_{k/m}^{k+1/m} |g| - \left| \int_{k/m}^{k+1/m} f \right| < \varepsilon$

$\to \int_{k/m}^{k+1/m} |f| - \left| \int_{k/m}^{k+1/m} f \right| < 2\varepsilon$ for large m

Hence $\int_{k/m}^{k+1/m} |f| = \left| \int_{k/m}^{k+1/m} f \right|$ for large m, hence

$$\lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \left| \int_{k/m}^{k+1/m} f \right| = \lim_{m\to\infty} \sum_{k=-m^2}^{m^2} \int_{k/m}^{k+1/m} |f| = \int_{-\infty}^{\infty} |f| = \|f\|_{L^1}$$

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