2008, Spring

Incomplete: 4.

Problem 1 (?)**.**

(i) **No**. Suppose the integral exists. Then by Fubini,

$$\int_E \frac{1}{x-y}\mathsf{d}m(x,y) = \int_0^1 \int_0^1 \frac{1}{x-y}\mathsf{d}x\mathsf{d}y = \int_0^1 \log\left(1-\frac{1}{y}\right)\mathsf{d}y$$

is well defined. But this is impossible since whenever y belongs to the measure-1 set $[0,1) \subset [0,1]$, we have $1-\frac{1}{y} < 0$ and so $\log\left(1-\frac{1}{y}\right)$ isn't even defined. □

(ii) **Yes**. The integrand is in $\mathsf{L}^+(E,m)$ so by Tonelli,

$$\int_E \frac{1}{x+y}\mathsf{d}m(x,y) = \int_0^1 \int_0^1 \frac{1}{x+y}\mathsf{d}x\mathsf{d}y = \int_0^1 \log\left(1+\frac{1}{y}\right)\mathsf{d}y = \log(4)$$

after a routine computation. □

Problem 2.

Let $\mathcal{S} := \{E \subset [0,1] \mid E \text{ compact and } \mu(E) = 1\}$. Firstly if $E_1, E_2 \in \mathcal{S}$, then certainly $E_1 \cup E_2 \subset [0,1]$; so $1 = \mu(E_1) \le \mu(E_1 \cup E_2) \le 1$, whereby

$$\mu(E_1 \cup E_2) = 1 \implies \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2) - \mu(E_1 \cup E_2) = 1 + 1 - 1 = 1.$$

Then inductively, any finite collection $\{E_j\}_{j=1}^m \subset \mathcal{S}$ has measure-1 intersection. We now claim that $\mu(K) = 1$, where K is the (potentially uncountable) intersection $\bigcap_{E\in\mathcal{S}} E$. To see this, let $U \subset [0,1]$ be an open set with $U \supset K$. Then the family of closed sets $\mathcal{T} := \{E \setminus U \mid E \in \mathcal{S}\}$ must satisfy $\bigcap_{E\setminus U \in \mathcal{T}}(E \setminus U) = $. This means that $\mathcal{T}$ doesn't have the finite intersection property, since any family of closed subsets of the compact space $[0,1]$ with this property has nonempty intersection. Thus there's a finite collection $\{E_j \setminus U\}_{j=1}^m \subset \mathcal{T}$ with empty intersection, giving

$$\bigcap_{j=1}^m (E_j \setminus U) = \implies \bigcap_{j=1}^m E_j \subset U \implies 1 = \mu\Big(\bigcap_{j=1}^m E_j\Big) \le \mu(U).$$

Since $U \supset K$ was an arbitrary open set, we have that

$$1 \le \mathsf{inf}\{\mu(U) \mid U \subset [0,1] \text{ open and } U \supset K\} = \mu(K) \le 1$$

by outer regularity of μ. Therefore $\mu(K) = 1$. □

Problem 3.

Neither implication holds.

- Let $f := \mathbb{1}_{(1/2,1]}$, which is continuous a.e. on $[0,1]$, and suppose that there's some continuous $g : [0,1] \to \mathbb{R}$ with $g = f$ a.e. For all $j \ge 3$, the sets $(1/2 - 1/j, 1/2), (1/2, 1/2 + 1/j)$ have positive measure, and thus contain some x_j, y_j, respectively, with $g(x_j) = f(x_j) = 0$ and $g(y_j) = f(y_j) = 1$. Moreover, $x_j \nearrow 1/2$ and $y_j \searrow 1/2$ as $j \to \infty$, so by continuity of g,

  $$g\left(\frac{1}{2}-\right) = \lim_{j\to\infty} g(x_j) = \lim_{j\to\infty} 0 = 0, \quad g\left(\frac{1}{2}+\right) = \lim_{j\to\infty} g(y_j) = \lim_{j\to\infty} 1 = 1,$$

  which is impossible since g is continuous at $1/2$. □

- Let $f := \mathbb{1}_{\mathbb{Q}\cap[0,1]}$ and $g :\equiv 0$. Then $f = 0 = g$ outside of the null set $\mathbb{Q} \cap [0,1]$, but f is nowhere continuous on $[0,1]$. □