Spring 07:

(1.) (μ_n) seq. of measures on $(X, \mathcal{M})$ with $\mu_1(E) \le \mu_2(E) \le \cdots$ for all $E \in \mathcal{M}$; Let $\mu(E) = \lim_{n\to\infty} \mu_n(E)$; show μ is a measure

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Clearly $\mu(\emptyset) = \lim_{n\to\infty} \mu_n(\emptyset) = \lim_{n\to\infty} 0 = 0$, hence must only verify countable additivity; let $A = \bigcup_{k=1}^{\infty} A_k$, $A_k \in \mathcal{M}$ disjoint

- Case $\mu(A) < \infty$: $\mu(A) = \lim_{n\to\infty} \mu_n(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k)$
$\le \sum_{k=1}^{\infty} \mu(A_k)$ since $\mu_n(A_k) < \mu(A_k)$ for all n.

Now choose $N < \infty$; Then:

$$\sum_{k=1}^{N} \mu(A_k) = \sum_{k=1}^{N} \lim_{n\to\infty} \mu_n(A_k) = \lim_{n\to\infty} \sum_{k=1}^{N} \mu_n(A_k) \le \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k)$$
$$= \lim_{n\to\infty} \mu_n\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right).$$

So we have $\sum_{k=1}^{N} \mu(A_k) \le \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$; letting $N \to \infty$, we get that $\sum_{k=1}^{\infty} \mu(A_k) \le \mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu(A)$; hence $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$

- Case $\mu(A) = \infty$: $\infty = \mu(A) = \lim_{n\to\infty} \mu_n(A)$
$$= \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k) \le \sum_{k=1}^{\infty} \mu(A_k) \Rightarrow \sum_{k=1}^{\infty} \mu(A_k) = \infty$$
$$\Rightarrow \mu(A) = \infty = \sum_{k=1}^{\infty} \mu(A_k).$$

Alternative: $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \mu_n\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k)$

$\mu_n(A_k) \le \mu_{n+1}(A_k)$ for all n, k, and all positive measures, hence apply MCT:

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \mu_n(A_k) = \sum_{k=1}^{\infty} \lim_{n\to\infty} \mu_n(A_k) = \sum_{k=1}^{\infty} \mu(A_k)$$

(2.) $\mu(X) < \infty$, $f \in L^1(\mu)$, $f > 0$, $0 < \alpha < \mu(X)$.

(a) Show $\inf\{\int_E f\,d\mu : \mu(E) \geq \alpha\} > 0$

(b) Show (a) is false with $\mu(X) = \infty$

(a) Let $E_n = \{f \geq \frac{1}{n}\}$; clearly $E_n \subseteq E_{n+1}$, and since $f > 0$,

$X = \bigcup_{n=1}^{\infty} E_n$; so by continuity from below, $\lim_{n\to\infty} \mu(E_n) = \mu(X)$.

Therefore, $\exists N$ such that $\mu(E_N) > \mu(X) - \alpha$

Now choose $E \subseteq X$ such that $\mu(E) \geq \alpha$ (note that N does not depend on the choice of E). So we have:

$$\mu(X) < \mu(E_N) + \alpha \leq \mu(E_N) + \mu(E)$$

hence $\mu(E \cap E_N) > 0$, and since $E \cap E_N \subseteq E$, we have now:

$$\int_E f\,d\mu \geq \int_{E \cap E_N} f\,d\mu \geq \frac{\mu(E \cap E_N)}{N}$$

But, $\mu(E \cap E_N) = \mu(E) + \mu(E_N) - \mu(E \cup E_N)$

$$> \alpha + (\mu(X) - \alpha) - \mu(X) = 0$$

Hence $\mu(E \cap E_N) > 0$, hence $\int_E f\,d\mu \geq \int_{E \cap E_N} f\,d\mu \geq \frac{\mu(E \cap E_N)}{N} > 0$,

so $\underline{\inf\{\int_E f\,d\mu : \mu(E) \geq \alpha\} > 0}$

(b) Let $X = [1, \infty)$; $f(x) = \frac{1}{x^2}$; $\alpha = 1$; $E_n = [n, n+\alpha]$

Then $\int_{E_n} f\,dm = \int_{E_n} \frac{dx}{x^2} = \frac{1}{n} - \frac{1}{n+\alpha} = \frac{\alpha}{n(n+\alpha)} \longrightarrow 0$ as $n \to \infty$,

Recall that $m(E_n) = \alpha$ for all n; thus:

$$\inf\{\int_E f\,d\mu : \mu(E) \geq \alpha\} \leq \inf\{\int_{E_n} f\,d\mu : E_n\} = 0$$

$$\Rightarrow \underline{\inf\{\int_E f\,d\mu : \mu(E) \geq \alpha\} = 0}$$

(3.) $f:[0,1]\to\mathbb{R}$ continuous, $G(f)=\{(x,f(x)) : x\in[0,1]\}$.
Show that $m(G(f))=0$

$f:[0,1]\to\mathbb{R}$ continuous and $[0,1]$ compact, hence f is uniformly continuous; i.e., for $\varepsilon>0$; $\exists\,\delta>0$ such that $|x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$ for all x, $y\in[0,1]$, hence we may take $\delta<1$

Now, let $n=\min\{n\in\mathbb{N} : n\delta>1\}$ and then let $0=x_0<x_1<\cdots<x_n=1$ in $[0,1]$ such that $|x_i-x_{i-1}|<\delta$

So for $z\in[x_{i-1},x_i]$, we have $|z-x_i|<\delta \Rightarrow |f(z)-f(x_i)|<\varepsilon$

$$\Rightarrow G(f)\subseteq\bigcup_{i=1}^{n}[x_{i-1},x_i]\times[f(x_i)-\varepsilon, f(x_i)+\varepsilon]$$

$$\Rightarrow m(G(f))\leq\sum_{i=1}^{n}m([x_{i-1},x_i]\times[f(x_i)-\varepsilon,f(x_i)+\varepsilon])$$
$$<\sum_{i=1}^{n}\delta\cdot 2\varepsilon = 2\varepsilon n\delta$$

Since n is the smallest s.t. $n\delta>1$, we must have that $(n-1)\delta\leq 1 \Rightarrow n\delta-\delta\leq 1 \Rightarrow n\delta\leq\delta+1$

Hence $2\varepsilon n\delta\leq 2\varepsilon(\delta+1)<2\varepsilon(2)=4\varepsilon$ since $\delta<1$.

Thus $\underline{m(G(f))<4\varepsilon}$; letting $\varepsilon\to 0$, we get our result.

(4.) $n \geq 1$; show that $g(u) = \int_{-\infty}^{\infty} \frac{x^n e^{ux}}{e^x+1} dx$ for $u \in (0,1)$.

Let $f(x,u) = \frac{x^n e^{ux}}{e^x+1}$; recall that the statement is true if $\left|\frac{\partial}{\partial u} f(x,u)\right| \leq g(x)$ some $g \in L^1$, for all x, u. Recall that differentiation is local, hence we may restrict to $u \in (0, 1-\varepsilon) \subsetneq (0,1)$.

For $u \in (0, 1-\varepsilon)$, we have:

for $e^x > 1$, $e^{ux} < e^{(1-\varepsilon)x}$ and for $e^x < 1$, $e^{ux} < 1$.

Hence for $u \in (0, 1-\varepsilon)$, $e^{ux} < 1 + e^{(1-\varepsilon)x}$

Then: $\left| \frac{x^{n+1} e^{ux}}{e^x+1} \right| \leq \left| \frac{x^{n+1}(e^{(1-\varepsilon)x} + 1)}{e^{x}+1} \right|$

and clearly $\frac{x^{n+1}}{e^x+1}$ is integrable, as is

$$\frac{x^{n+1} e^{(1-\varepsilon)x}}{e^x+1} = \frac{x^{n+1}}{e^{\varepsilon x}+1} \text{ integrable}$$

Hence we may take the derivative inside the integral:

$$g'(u) = \int_{-\infty}^{\infty} \frac{x^{n+1} e^{ux}}{e^x+1} dx$$