2007, Spring

Problem 1.

Firstly, $\mu() = \lim_{n\to\infty} \mu_n() = \lim_{n\to\infty} 0 = 0$. Now let $\{E_j\}_{j\in J} \subset \mathcal{M}$ be a disjoint collection indexed by a countable set $J \subset \mathbb{N}$, and for each $n \in \mathbb{N}$, let $f_n : \mathbb{N} \to \mathbb{R}$ be given by $f_n(j) := \mu_n(E_j)$. By assumption, $f_1 \le f_2 \le \cdots$, and $f_n \nearrow f$ for $f(j) := \mu(E_j)$. If ν is the counting measure on $\mathbb{N}$, then

$$\mu\Big(\bigcup_{j\in J} E_j\Big) = \lim_{n\to\infty} \mu_n\Big(\bigcup_{j\in J} E_j\Big) = \lim_{n\to\infty} \sum_{j\in J} \mu_n(E_j) = \lim_{n\to\infty} \int_{\mathbb{N}} f_n \mathsf{d}\nu = \int_{\mathbb{N}} f \mathsf{d}\nu = \sum_{j\in J} \mu(E_j)$$

by monotone convergence. □

Problem 2.

(a) Let $0 < \alpha < \mu(X)$, and assume the $\inf$ in question is 0. Then we can find a sequence $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ such that $\mu(E_j) \ge \alpha$ and $\int_X f \mathbb{1}_{E_j} = \int_{E_j} f < j^{-1}$. Then the sequence $\{f\mathbb{1}_{E_j}\}_{j=1}^{\infty}$ converges to 0 in measure, so there's some subsequence $\{f\mathbb{1}_{E_{j_k}}\}_{k=1}^{\infty}$ converging to 0 a.e. In this case,

$$0 = \mu\left(\limsup_{k\to\infty} E_{j_k}\right) = \mu\Big(\bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} E_{j_k}\Big)$$

so for any $\epsilon > 0$ there must be some $\ell \in \mathbb{N}$ satisfying the last inequality below,

$$\alpha \le \mu(E_{j_\ell}) \le \mu\Big(\bigcup_{k=\ell}^{\infty} E_{j_k}\Big) < \epsilon.$$

Choosing $\epsilon < \alpha$ gives a contradiction. □

(b) Let $X := (1,\infty)$ with Lebesgue measure μ. The function $f(x) := x^{-2}$ is strictly positive on $(1,\infty)$ and $\int_{(1,\infty)} f = 1$, so $f \in \mathsf{L}^1(\mu)$. However for $\alpha := 1$, the intervals $(j, j+1)$ for $j \in \mathbb{N}$ satisfy $\mu((j,j+1)) = 1$, and for any $\epsilon > 0$, we can choose j large enough so that

$$\int_{(j,j+1)} f = \int_j^{j+1} \frac{\mathsf{d}x}{x^2} = \frac{1}{j^2+j} < \epsilon.$$

Thus the $\inf$ in question must be 0. □

Problem 3.

Denote by μ the Lebesgue measure on $\mathbb{R}^2$, and let $\epsilon > 0$. Since $[0,1]$ is compact, f is uniformly continuous, so there's some $0 < \delta < 1$ so that $|f(x) - f(y)| < \epsilon/4$ whenever $|x-y| < \delta$. Let $0 = x_0 < x_1 < \cdots < x_{m-1} < x_m = 1$ be a partition with $|x_j - x_{j+1}| < \delta$ for each $0 \le j \le m-1$ and with $m \in \mathbb{N}$ the smallest integer satisfying $m\delta > 1$. Then $(m-1)\delta \le 1$ and so $m\delta \le 1 + \delta < 2$. Our choice of δ yields

$$\mathsf{graph}(f) \subset \bigcup_{j=0}^{m-1} [x_j, x_{j+1}] \times \left[f(x_j) - \frac{\epsilon}{4}, f(x_j) + \frac{\epsilon}{4}\right] \implies \mu(\mathsf{graph}(f)) \le \sum_{j=0}^{m-1} \delta \cdot \frac{2\epsilon}{4} = m\delta \cdot \frac{\epsilon}{2} < \epsilon.$$

Therefore $\mu(\mathsf{graph}(f)) = 0$. □

Problem 4 (?).

Fix $u \in (0,1)$. Provided we may exchange the order of differentiation and integration, then

$$g'(u) = \int_{-\infty}^{\infty} \frac{\mathsf{d}}{\mathsf{d}u}\left(\frac{x^n e^{ux}}{e^x+1}\right)\mathsf{d}x = \int_{-\infty}^{\infty} \frac{x^{n+1}e^{ux}}{e^x+1}\mathsf{d}x.$$

This exchange is valid if the integrand on the right-hand side is bounded (in magnitude) a.e. by an integrable function. To see this, let $\epsilon > 0$ be such that $u \in (0, 1-\epsilon)$. Then for $x > 0$, we have

$$1 < e^x \implies e^{ux} = (e^x)^u < (e^x)^{1-\epsilon} = e^{(1-\epsilon)x}$$

and for $x < 0$ we have $e^x < 1$. So for any $x \in \mathbb{R}$, we have $e^{ux} < 1 + e^{(1-\epsilon)x}$, whereby

$$\left|\frac{x^{n+1}e^{ux}}{e^x+1}\right| \le \left|\frac{x^{n+1}(1+e^{(1-\epsilon)x})}{e^x+1}\right| \le \left|\frac{x^{n+1}}{e^x+1}\right| + \left|\frac{x^{n+1}e^{(1-\epsilon)x}}{e^{x+1}}\right| \le \left|\frac{x^{n+1}}{e^x+1}\right| + \left|\frac{x^{n+1}}{e^{1+\epsilon x}}\right|.$$

Both summands on the right are integrable, so this completes the proof. □