Ob Sp. 1

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. For each of (i) and (ii) give a proof or a counterexample.

(i) Is it true that f is bounded on $\mathbb{R}$?

(ii) Is it true that $f(x) \to 0$ as $x \to \infty$?

How do the results for (i) and (ii) change under the additional assumption that f' exists everywhere and is bounded?

Counterexample for (i) and (ii):

Let $g = \sum_{n=1}^{\infty} 2^n \chi_{[n, n+1/2^{2n}]}$ For $K = 1, 2, 3, \ldots$

$g(K) = 2^K \to \infty$ as $K \to \infty$. Thus g is not bounded on $\mathbb{R}$ and $g(x) \not\to 0$ as $x \to \infty$.

And $\int |g| = \sum_{n=1}^{\infty} \int 2^n \chi_{[n, n+1/2^{2n}]} = \sum_{n=1}^{\infty} 2^n/2^{2n} = \sum_{n=1}^{\infty} 1/2^n < \infty$.

g is not continuous, but define f to be the function which has a triangle under each rectangle of the indicators in g, whose height is equal to that of the rectangle.

Then $\int |f| < \int |g| < \infty$, and f unbounded, $f \not\to 0$ as $x \to \infty$.

□

Now, assume f' exists everywhere and is bounded by M.

(ii) Assume not: then for some $\varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists\ x > N$ s.t. $|f(x)| > \varepsilon$.

Then we have $\int_x^{x+\varepsilon/M} |f(x)| \geq \varepsilon^2/2M$, since MVT will not allow $|f(x)|$ to dip below the triangle of area $\varepsilon^2/2M$.

We can find infinitely many of these situations that are non overlapping, since $f \not\to 0$. Thus $\int |f| \geq \sum^{\infty} \varepsilon^2/2M = \infty$. $\Rightarrow\Leftarrow$.

□

(i) By the same logic, $f(x) \to 0$ as $x \to -\infty$.

Thus, $\exists\ N \in \mathbb{N}$ s.t. $\forall\ x \notin [-N, N]$, $|f(x)| < 1$.

And f continuous $\Rightarrow$ f bounded on $[-N, N]$, compact.

Hence, f bounded on $\mathbb{R}$.

□

06 Sp. 2

(2) For $y > 0$ define
$$G(y) = \int_0^\infty \frac{1 - e^{-yx^2}}{x^2}\,dx.$$

(a) Show that this integral is finite for all $y > 0$.

(b) Show that G is differentiable, and find an explicit formula for $G'(y)$ and $G(y)$. HINT: You may take as given that $\int_0^\infty e^{-s^2}\,ds = \sqrt{\pi}/2$.

(a) We see that $0 < e^{-yx^2} \le 1 \implies \frac{1-e^{-yx^2}}{x^2} \le \frac{1}{x^2}$

Thus for $\varepsilon > 0$, $\int_\varepsilon^\infty \frac{1-e^{-yx^2}}{x^2}\,dx \le \int_\varepsilon^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_\varepsilon^\infty = \frac{1}{\varepsilon} < \infty.$

Now, by L'Hopital, we have
$$\lim_{x\to 0} \frac{1-e^{-yx^2}}{x^2} = \lim_{x\to 0} \frac{2yxe^{-yx^2}}{2x} = \lim_{x\to 0} ye^{-yx^2} = y$$

Thus $\exists\ \varepsilon > 0$ s.t. $\frac{1-e^{-yx^2}}{x^2} < 2y \quad \forall\ x \in (0, \varepsilon)$.

Hence, $\int_0^\infty \frac{1-e^{-yx^2}}{x^2}\,dx = \int_0^\varepsilon \frac{1-e^{-yx^2}}{x^2}\,dx + \int_\varepsilon^\infty \frac{1-e^{-yx^2}}{x^2}\,dx \le 2y\varepsilon + \frac{1}{\varepsilon} < \infty.$

$\square$

(b)
$$\begin{aligned}
G'(y) = \lim_{h\to 0} \frac{G(y+h) - G(y)}{h} &= \lim_{h\to 0} \frac{1}{h} \int_0^\infty \frac{1-e^{-(y+h)x^2}}{x^2} - \frac{1-e^{-yx^2}}{x^2}\,dx \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^\infty \frac{e^{-yx^2} - e^{-yx^2}e^{-hx^2}}{x^2}\,dx \\
&= \lim_{h\to 0} \frac{1}{h} \int_0^\infty e^{-yx^2} \frac{1-e^{-hx^2}}{x^2}\,dx \\
&= \int_0^\infty e^{-yx^2} \lim_{h\to 0} \frac{1-e^{-hx^2}}{hx^2}\,dx && \text{by DCT} \\
&= \int_0^\infty e^{-yx^2}\,dx && \text{by L'Hopital} \\
&= \frac{1}{\sqrt{y}} \int_0^\infty e^{-s^2}\,ds && s = \sqrt{y}\,x \\
&= \sqrt{\pi}/2\sqrt{y} && ds = \sqrt{y}\,dx
\end{aligned}$$

$\implies G(y) = \sqrt{\pi y} + C$. And $G(y) \to 0$ as $y \to 0$, by DCT

$\implies G(y) = \sqrt{\pi y}$.

$e^{-yx^2} \frac{1-e^{-hx^2}}{x^2} \le \frac{1-e^{-hx^2}}{x^2} \le \frac{1-e^{-(\sup h)x^2}}{x^2}$ integrable on $(0,\infty)$ by (a),

$\forall\ h$ in the sequence $\implies$ DCT applies.

$$\lim_{h\to 0} \frac{1-e^{-hx^2}}{hx^2} = \lim_{h\to 0} \frac{x^2 e^{-hx^2}}{x^2} = \lim_{h\to 0} e^{-hx^2} = 1.$$

$\square$

06 Sp. 3

(3) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f_n, f be real-valued measurable functions and suppose $f_n \to f$ a.e. Then there exists a partition of X into disjoint measurable sets $E_0, E_1, E_2, \ldots$ with $\mu(E_0) = 0$ and with $f_n \to f$ uniformly on E_i for each $i \geq 1$. HINT: Egoroff's Theorem requires a finite measure space.

$(X, \mathcal{M}, \mu)$ σ-finite $\Rightarrow$ $\exists$ countable partition $X = X_1 \sqcup X_2 \sqcup \cdots$ s.t. $\mu(X_i) < \infty$ $\forall i$. Thus Egoroff's Thm can be used on an arbitrary X_i. $\Rightarrow$ $\exists$ $F_{i,1} \subset X_i$ s.t. $\mu(X_i \setminus F_{i,1}) < 1$ and $f_n \to f$ uniformly on $F_{i,1}$.
Then we repeat: $\exists$ $F_{i,2} \subset X_i \setminus F_{i,1}$ s.t. $\mu(X_i \setminus \bigcup_{k=1}^{2} F_{i,k}) < \frac{1}{2}$ and $f_n \to f$ uniformly on $F_{i,2}$.
This continues, and for arbitrary j we see $\exists$ $F_{i,j} \subset X_i \setminus \bigcup_{k=1}^{j-1} F_{i,k}$ s.t. $\mu(X_i \setminus \bigcup_{k=1}^{j} F_{i,k}) < \frac{1}{j}$ and $f_n \to f$ uniformly on $F_{i,j}$.

Now, define $F_{i,0} := X_i \setminus \bigcup_{k=1}^{\infty} F_{i,k} = \bigcap_{j=1}^{\infty} (X_i \setminus \bigcup_{k=1}^{j} F_{i,k})$ which is a nested intersection. And since $\mu(X_i \setminus F_{i,1}) < \infty$, we can use continuity from above to deduce
$$\mu(F_{i,0}) = \lim_{j \to \infty} \mu(X_i \setminus \bigcup_{k=1}^{j} F_{i,k}) \leq \lim_{j\to\infty} \frac{1}{j} = 0$$
$\Rightarrow$ $\mu(F_{i,0}) = 0$.

Finally, define $E_0 := \bigcup_{i=1}^{\infty} F_{i,0}$, and define $\{E_i\}_1^{\infty}$ to be an enumeration of the countable $\{F_{i,k}\}_{i,k=1}^{\infty}$. Countable additivity gives $\mu(E_0) = 0$, and we've seen that $f_n \to f$ uniformly on each E_i, $i \geq 1$.

$\square$

06 Sp. 4

(4) A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *lower semi-continuous* if

$$\liminf g(x_n) \geq g(x) \quad \text{whenever } x_n \to x.$$

(a) Suppose that $f_k, k = 1, 2, 3, \ldots$ is a sequence of continuous functions, and $f(x) = \sup_{k \geq 1} f_k(x)$ is finite for all x. Show that f is lower semi-continuous.

(b) Show that a lower semi-continuous function is measurable.

(a) Lemma: If $g^{-1}((a,\infty))$ is open $\forall a \in \mathbb{R}$, then g is l.s.c.

Pf: let $x \in \mathbb{R}$. let $x_n \to x$. Assume $g(x) > \liminf g(x_n)$.
Set $a \in \mathbb{R}$ s.t. $\liminf g(x_n) < a < g(x)$.
$g^{-1}((a,\infty))$ open $\Rightarrow$ $g^{-1}((-\infty, a])$ closed.
$\liminf g(x_n) < a$ $\Rightarrow$ $\exists$ subsequence $\{x_{n_i}\}$ s.t. $\lim g(x_{n_i}) < a$
and s.t. $g(x_{n_i}) < a$ $\forall i$. Then $\{x_{n_i}\} \subset g^{-1}((-\infty, a])$
$\Rightarrow$ $\lim x_{n_i} = x \in g^{-1}((-\infty, a])$ $\Rightarrow$ $g(x) \leq a$ $\Rightarrow\Leftarrow$.

$\square$

Now, consider $f^{-1}((a,\infty))$. Let $x \in f^{-1}((a,\infty))$.
$\Rightarrow f(x) > a$ $\Rightarrow$ $\sup_k f_k(x) > a$ $\Rightarrow$ $\exists$ K s.t. $f_K(x) > a$,
f_K cont. $\Rightarrow$ $\exists$ δ s.t. $f_K(y) > a$ $\forall y \in (x-\delta, x+\delta)$ $\Rightarrow$ $f(y) > a$ $\forall$ such y
$\Rightarrow$ $(x-\delta, x+\delta) \subset f^{-1}((a,\infty))$. Thus, $f^{-1}((a,\infty))$ open $\Rightarrow$ f l.s.c. $\square$

(b) Claim: if g is l.s.c. $\Rightarrow$ $g^{-1}((-\infty, a])$ closed for $a \in \mathbb{R}$.
Pf: Let $x_n \to x$ in $\mathbb{R}$, with $\{x_n\} \subset g^{-1}((-\infty, a])$.
$\Rightarrow$ $g(x_n) \leq a$ $\forall$ n.
g l.s.c. $\Rightarrow$ $g(x) \leq \liminf g(x_n) \leq a$
$\Rightarrow$ $x \in g^{-1}((-\infty, a])$
Thus $g^{-1}((-\infty, a])$ closed, measurable

Hence, g is measurable

$\square$