Spring 06

① $f:\mathbb{R}\to\mathbb{R}$ continuous, L^1. Proof or counterex:

(i) f bdd

(ii) $f(x)\to 0$ as $x\to\infty$

what if f' bdd?

(i) False: Let $g(x) = \begin{cases} e^{\frac{-1}{1-x^2}}, & |x|<1 \\ 0, & |x|\geq 1 \end{cases}$; note that $e^{\frac{-1}{1-x^2}} \leq e^{-1}$ for $x\in(-1,1)$

So $\int |g(x)| = \int_{-1}^{1} e^{\frac{-1}{1-x^2}} = \int_{-1}^{1} \frac{1}{e^{\frac{1}{1-x^2}}} < \int_{-1}^{1} \frac{1}{e}\,dx < \infty$, so

Then let $g_n(x) = n\,g((x-n)n^3)$; then we have:

$y=(x-n)n^3$
$dy = n^3 dx$

$$\int g_n(x) = \int n\,g((x-n)n^3)\,dx = \int \frac{n\,g(y)\,dy}{n^3} = \frac{\|g\|}{n^2}$$

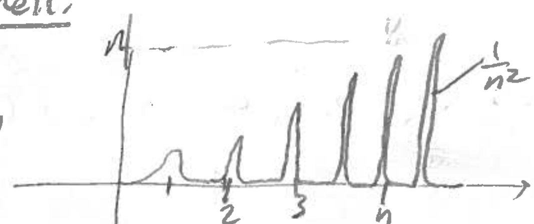
Now note that $g_n(n) = n\,g(0) = n e^{\frac{-1}{1}} = ne^{-1}$

and define $G(x) = \sum_{k\geq 3} g_k(x)$; then $G(n) = \sum_{k\geq 3} g_k(n) \geq ne^{-1}$, hence as $n\to\infty$, $G(n)\to\infty$, hence unbdd, but:

$$\int |G(x)| = \int \Big|\sum_{k\geq 3} g_k(x)\Big| \leq \int \sum_{k\geq 3} |g_k(x)| \underset{\text{Tonelli}}{=} \sum_{k\geq 3}\int |g_k(x)| = \sum_{k\geq 3} \frac{\|g\|}{n^2} < \infty,$$

hence $G\in L^1$.

(ii) False: Consider the function where $f(n)=n$, and



$f\equiv 0$ outside the intervals $\left(n-\frac{1}{2n^3}, n+\frac{1}{2n^3}\right)$, and $f(x)\leq n$ on.

Then: $\int_{n-\frac{1}{2n^3}}^{n+\frac{1}{2n^3}} f(x) \leq n\cdot\frac{1}{n^3} = \frac{1}{n^2}$, hence integrable, but $f\not\to 0$.

If $|f'(x)|\leq M$, by MVT, $|f(x)-f(y)|\leq M|x-y|$ for $x,y\in\mathbb{R}$,

hence f bdd

② $y>0$; $G(y) = \int_0^\infty \frac{1-e^{-yx^2}}{x^2}dx$

(a) Show $G(y)<\infty$ $\forall y>0$

(b) Find $G'(y)$

(a) See that $1-e^{-yx^2} = 1-\sum_{i=0}^{\infty}\frac{(-yx^2)^i}{i!} = -\sum_{i=1}^{\infty}\frac{(-1)^i y^i x^{2i}}{i!}$

$$\Rightarrow \frac{1-e^{-yx^2}}{x^2} = -\sum_{i=1}^{\infty}\frac{(-1)^i y^i x^{2i-2}}{i!} = -\sum_{i=0}^{\infty}\frac{(-1)^{i+1}y^{i+1}x^{2i}}{i!} = y\sum_{i=0}^{\infty}\frac{(-1)^i y^i x^{2i}}{i!} = ye^{-yx^2}$$

hence $\int_0^\infty \frac{1-e^{-yx^2}}{x^2}dx = \int_0^\infty ye^{-yx^2}dx < \infty$

(b) $G'(y) = \lim_{t\to y}\frac{1}{y-t}\int_0^\infty\left[\left(\frac{1-e^{-yx^2}}{x^2}\right)-\left(\frac{1-e^{-tx^2}}{x^2}\right)\right]dx$

$= \lim_{t\to y}\int_0^\infty \frac{e^{-tx^2}-e^{-yx^2}}{y-t}\cdot\frac{dx}{x^2} = \lim_{t\to y} -\int_0^\infty \frac{e^{-yx^2}-e^{-tx^2}}{y-t}\frac{dx}{x^2}$ (*) $f(x,t)$

$\longrightarrow$ Notice that $\lim_{t\to y}\frac{e^{-yx^2}-e^{-tx^2}}{y-t} = \frac{d}{dt}\left(e^{-tx^2}\right)\Big|_{t=y} = -x^2e^{-yx^2}$ (**)

Now, by the MVT, for some $t=c$:

$$\left|\frac{e^{-yx^2}-e^{-tx^2}}{y-t}\right| = \left|\frac{\partial}{\partial t}(f(x,t))_{t=c}\right| \le \sup_t\left|\frac{\partial}{\partial t}f(x,t)\right|$$

Notice that $\frac{\partial}{\partial t}f(x,t) = -x^2e^{-tx^2} = \frac{-x^2}{1+tx+\frac{t^2x^2}{2}+\frac{t^3x^3}{6}+\frac{t^4x^4}{24}+\cdots}$

$\le \frac{-x^2}{\frac{t^4x^4}{24}} = \frac{-24}{t^4x^2} < \frac{-24}{\varepsilon^4x^2}$ for $t\in(\varepsilon,1)$ $\subseteq(0,1)$

So $\left|\frac{e^{-yx^2}-e^{-tx^2}}{y-t}\right| \le \frac{-24}{\varepsilon^2x^2}$, so apply DCT:

$$(*) = -\int_0^\infty \lim_{t\to y}\frac{e^{-yx^2}-e^{-tx^2}}{y-t}\frac{dx}{x^2} \overset{\text{By (**)}}{=} -\int_0^\infty \frac{-\cancel{x^2}e^{-yx^2}}{\cancel{x^2}}dx = \int_0^\infty e^{-yx^2}dx$$

Let $x=\frac{u}{\sqrt{y}}$; then $\int_0^\infty e^{-yx^2}dx = \int_0^\infty \frac{e^{-u^2}du}{\sqrt{y}} = \frac{\sqrt{\pi}}{2\sqrt{y}} = \frac{1}{2}\sqrt{\frac{\pi}{y}}$

(3.) $(X, \mathcal{M}, \mu)$ σ-finite, f_n, f $\mathbb{R}$-valued measurable

Suppose $f_n \to f$ a.e.. Show $\exists$ partition of X into disjoint meas. sets $E_0, E_1, \ldots$ with $\mu(E_0) = 0$ and $f_n \to f$ unif on E_i, $i \geq 1$

Since X is σ-finite, we have $X = \bigcup_{i=1}^{\infty} X_i$ for $\mu(X_i) < \infty$.

Let $A_n = \bigcup_{i=1}^{n} X_i \Rightarrow \lim_{n\to\infty} \mu(A_n) = \mu(X)$ by continuity from below since $A_n \subseteq A_{n+1}$. $(A_0 = \phi)$

Let $B_n = A_{n+1} \setminus A_n$, $B_0 = A_0 = \phi$; then the B_n are disjoint and $\mu(X) = \mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n) \Rightarrow \lim_{n\to\infty} \mu(B_n) = 0$

By Egoroff's theorem, since $\mu(B_n) < \infty$, B_n, for each $\frac{\varepsilon}{2^n}$, $\exists F_n \subseteq B_n$ such that $\mu(F_n) < \frac{\varepsilon}{2^n}$ and $f_n \to f$ unif on $B_n \setminus F_n$.

Now, $\mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$

Since the B_n are disjoint, we have that $\bigcup_{n=1}^{\infty} B_n \setminus F_n = \bigcup_{n=1}^{\infty} B_n \setminus \bigcup_{n=1}^{\infty} F_n$,

hence: $\mu(\bigcup B_n \setminus F_n) = \mu(\bigcup B_n \setminus \bigcup F_n) = \mu(\bigcup B_n) - \mu(\bigcup F_n) > \mu(X) - \varepsilon$

letting $\varepsilon \to 0$; we have $\mu(\bigcup B_n \setminus F_n) \geq \mu(X)$, hence $\mu(\bigcup B_n \setminus F_n) = \mu(X)$; and $\mu(\bigcup_{n=1}^{\infty} F_n) = 0$

Hence let $E_n = B_n \setminus F_n$ for $n \geq 1$ and $E_0 = \bigcup F_n$.

Then $E_0 \cup (\bigcup_{n=1}^{\infty} E_n) = X$

(4) $g: \mathbb{R} \to \mathbb{R}$ is l.s.c. if $\liminf_{n\to\infty} g(x_n) \geq g(x)$ for $x_n \to x$.

(a) f_k continuous, $f(x) = \sup_k f_k(x) < \infty$. Show f l.s.c.

(b) Show l.s.c. fn. is measurable.

(a) $\liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} \inf_{k\geq n} f(x_k) = \lim_{n\to\infty}\left(\inf_{k\geq n}\left(\sup_{j\geq 0} f_j(x_k)\right)\right)$

$\geq \lim_{n\to\infty}\left(\inf_{k\geq n} f_j(x_k)\right)$ for all j

$= \liminf_{n\to\infty} f_j(x_n)$ for all j

$= f_j(x)$ for all j

$\Rightarrow \sup_{j\geq 0}\left(\liminf_{n\to\infty} f(x_n)\right) \geq \sup_{j\geq 0} f_j(x) \Rightarrow \liminf_{n\to\infty} f(x_n) \geq f(x)$

(b) Consider the sets $f^{-1}([a,\infty)) = \{f \geq a\}$; f is Borel measurable if each of these sets is Borel measurable.

$\longrightarrow$ [L.S.C. at x if for $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow f(y) \geq f(x) - \varepsilon$]

Consider $x \in f^{-1}([a,\infty))$ and choose $\varepsilon > 0$.
Since f is l.s.c., $\exists \delta > 0$ s.t. $|x-y| < \delta \Rightarrow f(y) \geq f(x) - \varepsilon$,
here $y \in (x-\delta, x+\delta) \Rightarrow f(y) \geq f(x) - \varepsilon \geq a - \varepsilon$
Letting $\varepsilon \to 0$, $\exists \delta' > 0$ s.t. $y \in (x-\delta', x+\delta') \Rightarrow f(y) \geq a$, hence
$(x-\delta', x+\delta') \subseteq f^{-1}([a,\infty))$.
Hence an arbitrary $x \in f^{-1}([a,\infty))$ has an open nbhd contained in $f^{-1}([a,\infty))$, hence $f^{-1}([a,\infty))$ is open, hence Borel.

$\longrightarrow$ So f is measurable