2006, Spring

Problem 1.

- **No**. Consider the function $f : \mathbb{R} \to \mathbb{R}$ consisting of symmetric triangular spikes of height j and base $2j^{-3}$ at each integer $j \geq 2$ along $\mathbb{R}$. Explicitly, f is given by

$$f(x) := \begin{cases} j^4(x-j) & j \geq 2, x \in [j, j+j^{-3}), \\ j^4[(j+2j^{-3})-x] & j \geq 2, x \in [j+j^{-3}, j+2j^{-3}), \\ 0 & \text{else.} \end{cases}$$

The $\mathsf{L}^1(\mathbb{R})$-norm of f is given by the sum of the areas of the triangles,

$$\|f\|_{\mathsf{L}^1(\mathbb{R})} = \sum_{j=2}^{\infty} j \cdot \frac{1}{j^3} = \sum_{j=2}^{\infty} \frac{1}{j^2} < \infty.$$

However, f isn't bounded and $\lim_{x\to\infty} f(x)$ is nonexistent, so neither (i) nor (ii) hold. □

- **Both (i) and (ii) hold** if f' exists everywhere and $|f'| \leq C$ for some $C > 0$.

Assume first that $f(x) \not\to 0$ as $x \to \infty$. Then there's some $\epsilon > 0$ for which we can find a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}$ with $x_j \to \infty$ and $f(x_j) \geq \epsilon$ for each $j \in \mathbb{N}$. We may assume w.l.o.g. that $x_1 \leq x_2 \leq \cdots$ and $|x_{j+1} - x_j| > 2\epsilon/C$ for all $j \in \mathbb{N}$. Fix some $j \in \mathbb{N}$; then $|f(x_j)| \geq \epsilon$, so assume w.l.o.g. that $f(x_j) \geq \epsilon$. For any $y \in (x_j - (\epsilon/C), x_j)$, we have by the mean value theorem that

$$\frac{f(x_j) - f(y)}{x - y} \leq C \implies \epsilon \leq f(x_j) \leq C(x_j - y) + f(y) \implies C(y - x_j) + \epsilon \leq f(y),$$

and similarly $C(x_j - y) + \epsilon \leq f(y)$ for any $y \in (x_j, x_j + (\epsilon/C))$. Then

$$\int_{x_j-(\epsilon/C)}^{x_j+(\epsilon/C)} f(y)\mathsf{d}y \geq \int_{x_j-(\epsilon/C)}^{x_j} [C(y-x_j)+\epsilon]\mathsf{d}y + \int_{x_j}^{x_j+(\epsilon/C)} [C(x_j-y)+\epsilon]\mathsf{d}y = \frac{2\epsilon^2}{C},$$

and so

$$\int_{\mathbb{R}} |f| \geq \sum_{j=1}^{\infty} \int_{x_j-(\epsilon/C)}^{x_j+(\epsilon/C)} |f(y)|\mathsf{d}y \geq \sum_{j=1}^{\infty} \frac{2\epsilon^2}{C} = \infty,$$

contradicting $f \in \mathsf{L}^1(\mathbb{R})$.

Assume next that f is unbounded. If $f(x) \to 0$ as $x \to \infty$, then there some $M > 0$ large enough so that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$ with $|x| > M$. Thus f must be unbounded on the compact set $[-M, M]$, which is impossible since f is continuous. Hence $f(x) \not\to 0$ as $x \to \infty$, which leads to a contradiction as above. □

Problem 2.

(a) For any $x, y > 0$,

$$\frac{1-e^{-yx^2}}{x^2} = \frac{1}{x^2}\left[1 - \sum_{j=0}^{\infty} \frac{(-1)^j y^j x^{2j}}{j!}\right] = -\sum_{j=1}^{\infty} \frac{(-1)^j y^j x^{2(j-1)}}{j!} = -\sum_{j=0}^{\infty} \frac{(-1)^{j+1} y^{j+1} x^{2j}}{(j+1)!}$$

$$\leq y \sum_{j=0}^{\infty} \frac{(-1)^j y^j x^{2j}}{j!} = ye^{-yx^2}$$

and hence by the substitution $s := \sqrt{y}x$,

$$0 \le G(y) \le \int_0^\infty y e^{-yx^2} \mathrm{d}x = \sqrt{y} \int_0^\infty e^{-s^2} \mathrm{d}s = \frac{\sqrt{\pi y}}{2} < \infty.$$

□

(b) For any $y > 0$,

$$G'(y) = \lim_{z\to y} \frac{G(y) - G(z)}{y - z} = \lim_{z\to y} \int_0^\infty \frac{-e^{-yx^2} + e^{-zx^2}}{(y-z)x^2} \mathrm{d}x = -\lim_{z\to y} \int_0^\infty \frac{e^{-yx^2} - e^{-zx^2}}{y - z} \cdot \frac{1}{x^2} \mathrm{d}x.$$

Provided that we can justify moving the limit inside the integral, then

$$G'(y) = -\int_0^\infty \lim_{z\to y} \frac{e^{-yx^2} - e^{-zx^2}}{y - z} \frac{\mathrm{d}x}{x^2} = \int_0^\infty \frac{\mathrm{d}e^{-zx^2}}{\mathrm{d}z}\Big|_{z=y} \frac{\mathrm{d}x}{x^2} = \int_0^\infty \frac{-x^2 e^{-yx^2}}{x^2} \mathrm{d}x = \int_0^\infty e^{-yx^2} \mathrm{d}x,$$

and by the substitution $s := \sqrt{y}x$,

$$G'(y) = \frac{1}{\sqrt{y}} \int_0^\infty e^{-s^2} \mathrm{d}s = \frac{1}{2}\sqrt{\frac{\pi}{y}},$$

and taking the antiderivative gives $G(y) = \sqrt{\pi y} + c$ for some $c \in \mathbb{R}$. From the definition of G we see that $G(0) = 0$ and now that $G(0) = c$, whereby $c = 0$ and so $G(y) = \sqrt{\pi y}$. To justify exchanging the limit and integration above, it suffices by dominated convergence to bound the integrand by an integrable function. Assume w.l.o.g. that $y < z$. By the mean value theorem, there's some $z_0 \in (y, z)$ with

$$\left|\frac{-e^{-yx^2} + e^{-zx^2}}{(y-z)x^2}\right| = \left|\frac{\partial e^{-zx^2}}{\partial z}\Big|_{z=z_0} \cdot \frac{1}{x^2}\right| \le \sup_{z_1\in(y,z)} \left|\frac{\partial e^{-zx^2}}{\partial z}\Big|_{z=z_1} \cdot \frac{1}{x^2}\right| = \sup_{z_1\in(y,z)} \left|\frac{-x^2 e^{-z_1 x^2}}{x^2}\right|$$
$$= \sup_{z_1\in(y,z)} \left|1 + z_1 x + \frac{(z_1 x)^2}{2!} + \frac{(z_1 x)^3}{3!} + \frac{(z_1 x)^4}{4!} + \cdots\right|^{-1} \le \sup_{z_1\in(y,z)} \frac{2}{z_1^2 x^2} \le \frac{2}{y^2 x^2},$$

and the right-hand side, when regarded as a function of x on $(0, \infty)$, is integrable. □

Problem 3.

Since $(X, \mathcal{M}, \mu)$ is σ-finite, then $X = \bigsqcup_{j\in J} X_j$ for some countable collection $\{X_j\}_{j\in J} \subset \mathcal{M}$ with $\mu(X_j) < \infty$ for each $j \in J$. Fix some $j \in J$. By Egoroff, for each $k \in \mathbb{N}$, there's a subset $Y_{j,k} \subset X_j$ in $\mathcal{M}$ with $\mu(X_j \setminus Y_{j,k}) < k^{-1}$ and with $f_n \to f$ uniformly on $Y_{j,k}$. We may assume w.l.o.g. that $Y_{j,1} \subset Y_{j,2} \subset \cdots$, so by construction, $Y_{j,k} \nearrow X_j$ (up to a null set) as $k \to \infty$. Setting $F_{j,k} := Y_{j,k} \setminus Y_{j,k-1}$ for each $k \in \mathbb{N}$, we still have $f_n \to f$ uniformly on $F_{j,k}$, and furthermore the collection $\{F_{j,k}\}_{k\in\mathbb{N}}$ is disjoint, so we may write X as the disjoint union

$$X = E_0 \sqcup \bigsqcup_{\substack{j\in J\\ k\in\mathbb{N}}} F_{j,k},$$

where E_0 is the null set $\bigcap_{k=1}^\infty \bigcup_{j\in J}(X_j \setminus Y_{j,k})$. Letting $\{E_\ell\}_{\ell=1}^\infty$ be an enumeration of the countable collection $\{F_{j,k}\}_{j\in J, k\in\mathbb{N}}$, we obtain the desired partition. □

Problem 4.

(a) An equivalent definition for a function $g : \mathbb{R} \to \mathbb{R}$ to be l.s.c. is that $\{x \in \mathbb{R} \mid a < f(x)\}$ is an open set for all $a \in \mathbb{R}$ (see (b)). To see that f has this property, let $a \in \mathbb{R}$ and suppose $a < f(x) = \mathsf{sup}_{j \in \mathbb{N}} f_j(x)$ for some $x \in \mathbb{R}$. Then by definition of sup, there's some $k \in \mathbb{N}$ with $a < f_k(x)$. But f_k is continuous, so there's some $\delta > 0$ such that for all $y \in \mathbb{R}$ with $|x - y| < \delta$, we have $a < f_k(y) \leq \mathsf{sup}_{j \in \mathbb{N}} f_j(y) = f(y)$.

(Note that we in fact only need the f_j's to be l.s.c.) □

(b) This is very similar to problem 1 of 2010, Spring.