REAL ANALYSIS GRADUATE EXAM
Spring 2006

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and in $L^1(\mathbb{R})$. For each of (i) and (ii) give a proof or a counterexample.

(i) Is it true that f is bounded on $\mathbb{R}$?

(ii) Is it true that $f(x) \to 0$ as $x \to \infty$?

How do the results for (i) and (ii) change under the additional assumption that f' exists everywhere and is bounded?

(2) For $y > 0$ define

$$G(y) = \int_0^\infty \frac{1 - e^{-yx^2}}{x^2}\, dx.$$

(a) Show that this integral is finite for all $y > 0$.

(b) Show that G is differentiable, and find an explicit formula for $G'(y)$ and $G(y)$. HINT: You may take as given that $\int_0^\infty e^{-s^2}\, ds = \sqrt{\pi}/2$.

(3) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. Let f_n, f be real-valued measurable functions and suppose $f_n \to f$ a.e. Then there exists a partition of X into disjoint measurable sets $E_0, E_1, E_2, \ldots$ with $\mu(E_0) = 0$ and with $f_n \to f$ uniformly on E_i for each $i \geq 1$. HINT: Egoroff's Theorem requires a finite measure space.

(4) A function $g : \mathbb{R} \to \mathbb{R}$ is said to be *lower semi-continuous* if

$$\liminf g(x_n) \geq g(x) \quad \text{whenever } x_n \to x.$$

(a) Suppose that $f_k, k = 1, 2, 3, \ldots$ is a sequence of continuous functions, and $f(x) = \sup_{k\geq 1} f_k(x)$ is finite for all x. Show that f is lower semi-continuous.

(b) Show that a lower semi-continuous function is measurable.