Spring 05:

(1.) $f:[a,b]\to\mathbb{R}$ Hölder cont of order α if $\exists L$ s.t.

$$|f(x)-f(y)|\le L|x-y|^{\alpha} \quad \text{for all } x,y\in[a,b].$$

(i) Show f Hölder cont order $1 \Rightarrow$ Bounded variation

$|f(x)-f(y)|\le L|x-y|$ for all $x,y\in[a,b]$; Consider (x_i) partition of $[a,b]$:

Then $$\sum_{j=1}^{n}|f(x_j)-f(x_{j-1})|\le L\sum_{j=1}^{n}|x_j-x_{j-1}| = L\mu([a,b]) = L(b-a)$$

Taking the supremum over partitions, we have:

$$T_a^b(x) = \sup_{P=\{x\}}\Big\{\sum_{j=1}^{n}|f(x_j)-f(x_{j-1})|\Big\}\le L(b-a).$$

hence bounded variation

(ii) + (iii): • Find f s.t. f Hölder cont of order $\alpha\in(0,1)$ and $f\notin BV$

• Find f s.t. $f\in BV$ but f not Hölder cont for any α.

(2.) $f \in L'([0,1])$. For $k = 1, 2, \dots$ define
$$f_k(x) = k \int_{j/k}^{j+1/k} f(t)\,dt \quad \text{for } \frac{j}{k} \le x \le \frac{j+1}{k}$$
for $j = 0, \dots, k-1$.
Show $f_k \to f$ in L'

See that, if f is continuous, we have:
$$\int_0^1 |f_k(x) - f(x)|\,dx \le \sum_{0}^{k-1} \int_{j/k}^{j+1/k} |f_k(x) - f(x)|\,dx = \int_{j/k}^{j+1/k} \left| \left(k \int_{j/k}^{j+1/k} f(t)\,dt\right) - f(x) \right|$$
$$= k \int_{j/k}^{j+1/k} \left| \left(\int_{j/k}^{j+1/k} f(t)\,dt\right) - \frac{f(x)}{k} \right| dx$$
$$= k \int_{j/k}^{j+1/k} \left| \int_{j/k}^{j+1/k} (f(t) - f(x))\,dt \right| dx$$
$$\le k \int_{j/k}^{j+1/k} \int_{j/k}^{j+1/k} |f(t) - f(x)|\,dt\,dx.$$
now for $\varepsilon > 0$, $x \in [0,1]$, $\exists k$ s.t. $|t - x| < \frac{1}{k} \Rightarrow |f(t) - f(x)| < \varepsilon$

Hence:
$$< k \int_{j/k}^{j+1/k} \int_{j/k}^{j+1/k} \varepsilon\,dt\,dx = k \int_{j/k}^{j+1/k} \frac{\varepsilon}{k}\,dx$$
$$= \frac{\varepsilon}{k} \longrightarrow 0 \text{ as } k \to \infty.$$

Hence, since cont fns are dense in L', $\exists$ cont g s.t. $\int_0^1 |f - g| < \varepsilon$ and $\int_0^1 |f_k - g_k| < \varepsilon$ for each k. (since $f_k \in L'$ as well),

hence:
$$\int_0^1 |f_k - f| = \int_0^1 |f_k - g_k + g_k - g + g - f|$$
$$\le \int_0^1 |f_k - g_k| + |f - g| + |g_k - g|\,dx$$
$$< \varepsilon + \varepsilon + \int |g_k - g| = 2\varepsilon$$
for large k;
hence let $\varepsilon \to 0$ and result follows

$$\int |f_k(x)| \le \int_{j/k}^{j+1/k} |f_k(x)|$$
$$= \int_{j/k}^{j+1/k} \left| k \int_{j/k}^{j+1/k} f_k(t)\,dt \right| dx$$
$$= \left| k \int_{j/k}^{j+1/k} f_k(t)\,dt \right| \cdot \frac{1}{k}$$
$$= \left| \int_{j/k}^{j+1/k} f_k(x)\,dt \right| \le \int_{j/k}^{j+1/k} |f_k(x)|\,dt < \infty$$

(3.) (f_n), f $\mathbb{C}$-valued measurable on $(X, \mathcal{M}, \mu)$
Show that if f_n Cauchy in measure, then $\exists$ subseq $f_{n_k} \to g$ a.e. and $f_n \to g$ in measure

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→ Recall that f_n Cauchy in measure if for $\varepsilon > 0$,

$$\mu\{|f_n - f_m| \geq \varepsilon\} \longrightarrow 0 \text{ as } m, n \longrightarrow \infty.$$

So choose subsequence $g_j = f_{n_j}$ of f_n such that

$$\mu\{|g_j - g_{j+1}| \geq 2^{-j}\} \leq 2^{-j}$$

Let $F_k = \bigcup_{j=k}^{\infty} \{|g_j - g_{j+1}| \geq 2^{-j}\} \Rightarrow \mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}$

→ Now, if $x \notin F_k$, then for $i, j \geq k$ we have

$$(*) \quad |g_j(x) - g_i(x)| \leq \sum_{l=j}^{i-1} |g_{l+1}(x) - g_l(x)| \leq \sum_{l=j}^{i-1} 2^{-l} \leq 2^{1-j}$$

Hence g_i is pointwise Cauchy on F_k^c

→ Let $F = \bigcap_{k=1}^{\infty} F_k$; then $\mu(F) \leq \mu(F_k)$ for all k, hence $\mu(F) \leq 2^{1-k}$ for all k, hence $\mu(F) = 0$.

Then let $f(x) = \lim_{j \to \infty} g_j(x)$ for $x \notin F$, and $f(x) = 0$ for $x \in F$.
Hence $g_j \to f$ a.e. (on F^c, which is full measure since F is null)

Inequality $(*)$ gives us $|g_j(x) - f(x)| \leq 2^{1-j}$ for $x \notin F_k$, $j \geq k$,
hence for $x \in F_k$, $|g_j(x) - f(x)| \geq 2^{1-j}$, hence:

$$\mu\{|g_j - f| \geq 2^{1-j}\} \leq \mu(F_k) \longrightarrow 0 \text{ as } k \to \infty.$$

hence $g_j \to f$ in measure. (2)

But now, $\{|f_n - f| \geq \varepsilon\} \subseteq \{|f_n - g_j| \geq \frac{\varepsilon}{2}\} \cup \{|g_j - f| \geq \frac{1}{2}\varepsilon\}$
and $\mu\{|g_j - f| \geq \frac{1}{2}\varepsilon\} \to 0$ since $g_j \to f$ in measure and
$\mu\{|f_n - g_j| \geq \frac{\varepsilon}{2}\} = \mu\{|f_n - f_{n_j}| \geq \frac{\varepsilon}{2}\} \to 0$ since Cauchy in meas
→ so $f_n \to f$ in meas

4. Show $\sum_{i=1}^{\infty} a_i = \infty \iff \exists \{r_1, \dots\} = \mathbb{Q}$ s.t. $\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}$

($\Leftarrow$) Suppose $\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}$.

$m((r_i - a_i, r_i + a_i)) = 2a_i$

Then: $\infty = m(\mathbb{R}) = m\left(\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i)\right) \leq \sum m(r_i - a_i, r_i + a_i)$
$= \sum 2a_i$

$\Rightarrow \sum_{i=1}^{\infty} a_i = \infty$

($\Rightarrow$) Suppose $\sum a_i = \infty$:

First, if $\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}$, then $m\left(\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i)\right)$ must be infinite; but:

$$m\left(\bigcup_{1}^{\infty} (r_i - a_i, r_i + a_i)\right) \leq \sum m(r_i - a_i, r_i + a_i) = \sum 2a_i$$

hence the measure may be infinite since $\sum a_i = \infty$

$\to$ Now, we want to show that, for each $x \in \mathbb{R}$, $\exists i$ s.t. $x \in (r_i - a_i, r_i + a_i)$:

Since rationals are dense, for $x \in \mathbb{R}$, $\exists r_i$ such that $|x - r_i| < a_i$, hence $x \in (r_i - a_i, r_i + a_i)$

So $\bigcup_i (r_i - a_i, r_i + a_i) \supseteq \mathbb{R}$, hence $=$.