REAL ANALYSIS GRADUATE EXAM

SPRING 2005

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. A function $f : [a, b] \to \mathbb{R}$ is said to be Hölder continuous of order α if there is a constant L such that

$$|f(x) - f(y)| \leq L|x - y|^{\alpha} \qquad \text{for all } x, y \in [a, b].$$

(i) Show that if f is Hölder continuous of order 1 then it has bounded variation.

(ii) Let $\alpha \in (0, 1)$. Give an example of a function which is Hölder continuous of order α and does not have bounded variation.

(iii) Give an example of a function of bounded variation which is not Hölder continuous for any $\alpha > 0$.

2. Let $f \in L^1([0, 1])$. For $k = 1, 2, \ldots$ let f_k be the step function defined on $[0, 1]$ by

$$f_k(x) = k \int_{j/k}^{(j+1)/k} f(t)\, dt \quad \text{for } \frac{j}{k} \leq x < \frac{j+1}{k}.$$

Show that f_k tends to f in L^1 as $k \to \infty$.
(Hint: Treat first the case when f is continuous, and use approximation.)

3. Suppose that f_n, $n = 1, 2, \ldots$, and f are complex valued measurable functions on a measure space $(X, \mathcal{M}, \mu)$. Define the terms

(i) f_n converges in measure to f,

(ii) f_n is Cauchy in measure.

Show that if f_n is Cauchy in measure there exists a subsequence n_k and a measurable function g such that f_{n_k} converges to g almost everywhere, and f_n converges to g in measure.

4. Let $a_1, a_2, \ldots > 0$. Prove that $\sum_{i=1}^{\infty} a_i = \infty$ is a necessary and sufficient condition that there exists an enumeration of the rationals

$$\mathbb{Q} = \{r_1, r_2, \ldots\}$$

such that

$$\bigcup_{i=1}^{\infty} (r_i - a_i, r_i + a_i) = \mathbb{R}.$$