Spring 04 :

① $\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+\frac{1}{n}) - \cos(x)}{x^{3/2}} dx$ exists

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$= \lim_{n\to\infty} \int_0^1 n \chi_{[\frac{1}{n},1]}(x) \left(\frac{\cos(x+\frac{1}{n}) - \cos(x)}{x^{3/2}} \right) dx$

(*) Hence if $\left| \frac{\cos(x+\frac{1}{n}) - \cos(x)}{\frac{1}{n}} \right| \le |g(x)|$ for $g \in L^1$, $x \in [\frac{1}{n}, 1]$, for all n, then we may apply DCT:

$= \int_0^1 \lim_{n\to\infty} \chi_{[\frac{1}{n},1]}(x) \left(\frac{\cos(x+\frac{1}{n}) - \cos(x)}{\frac{1}{n} x^{3/2}} \right) dx$

$= \int_0^1 \frac{-\sin(x)}{x^{3/2}} dx$; now, $\frac{\sin(x)}{x}$ is bdd on $[0,1]$,

hence $\le \int_0^1 -\frac{1}{x^{1/2}} dx$, which is integrable on $[0,1]$.

So, need to show (*):

See that, by the MVT,

$\left| \chi_{[\frac{1}{n},1)} \frac{\cos(x+\frac{1}{n}) - \cos(x)}{\frac{1}{n}} \right| \le |\sin(\theta_n)|$ for some $\theta_n \in [x, x+\frac{1}{n}]$

Note that $\sin(x)$ is increasing on $[\frac{1}{n}, 1]$, (approximately linearly)

hence $|\sin\theta_n| \le |\sin(x+\frac{1}{n})| \le \sin(x) + \frac{1}{n}$; and since increasing on $[\frac{1}{n},1]$, $\sin(x)$ is minimal at $\frac{1}{n}$. Since $\frac{1}{n}$ is close to 0, $\sin(x)$ is approximately linear, hence $\sin(x_1) \ge \frac{1}{2}(\frac{1}{n})$, i.e. $2\sin(x) \ge \frac{1}{n}$, hence $|\sin\theta_n| \le |3\sin(x)|$,

which is integrable, and the statement follows from above

(2.) $(f_n) \subseteq L^+(\mu)$, $f_n \to f$ a.e., $\int_X f_n \, d\mu \to \int_X f \, d\mu$ as $n \to \infty$.

Show: $\int_X f_n g \, d\mu \to \int_X f g \, d\mu$ for g bdd, measurable.

Since $|f_n - f| \le |f_n| + |f|$, consider the sequence of positive measurable functions $g_n = |f_n| + |f| - |f_n - f|$.

Clearly $g_n \to 2|f|$ a.e.; now apply Fatou:

$$\int \liminf_{n\to\infty} g_n \le \liminf_{n\to\infty} \int g_n$$

$$\Rightarrow \int 2|f| \le \liminf_{n\to\infty} \int |f_n| + |f| - |f_n - f|$$

$$= \liminf_{n\to\infty} \left(\int |f_n| + \int |f| - \int |f_n - f| \right)$$

$$= 2\int |f| - \liminf_{n\to\infty} \int |f_n - f|$$

$\Rightarrow \liminf_{n\to\infty} \int |f_n - f| \le 0$, hence $= 0$, hence $f_n \to f$ in L^1.

Now; for g meas and $|g(x)| \le M$ bdd:

$$\left| \int g f_n - \int g f \right| \le \int |g f_n - g f| \le \int |g| \, |f_n - f| \le M \int |f_n - f|$$

and since $f_n \to f$, for $\frac{\varepsilon}{M} > 0$, $\exists N$ s.t. $n \ge N \Rightarrow \int |f_n - f| < \frac{\varepsilon}{M}$

hence for $n \ge N$, $\left| \int g f_n - \int g f \right| < \varepsilon$, hence $\lim_{n\to\infty} \int g f_n = \int g f$.

③ $f \in L^1(\mathbb{R}^d)$. Evaluate $\lim_{y\to\infty} \int_{\mathbb{R}^d} |f(x+y)-f(x)|\,dx$

Suppose ϕ is an integrable simple function, i.e. $\phi(x)=\sum a_i \chi_{E_i}(x)$.
Now see the following:

$$\int |\textstyle\sum a_i \chi_{E_i}(x+y) - \sum a_i \chi_{E_i}(x)| = \int |\sum a_i \chi_{E_i-y}(x) - \sum a_i \chi_{E_i}(x)|$$

For y large enough, we have $E_i - y \cap E_i = \phi$

$$= \int |\textstyle\sum a_i \chi_{E_i-y}(x)| + |\sum a_i \chi_{E_i}(x)|$$

$$= \int |\textstyle\sum a_i \chi_{E_i-y}(x)| + \int |\sum a_i \chi_{E_i}(x)|$$

$$= \int |\phi(x+y)| + \int |\phi(x)$$

Since Lebesgue int. is translation invariant

$$= 2\int |\phi(x)|$$

Now, since f is measurable, $\exists$ sequence (ϕ_n) of int. simple functions s.t. $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |f|$, $\phi_n \to f$ pointwise.

Then $|\phi_n(x+y)-\phi_n(x)| \longrightarrow |f(x+y)-f(x)|$ pointwise, and!

$|\phi_n(x+y)-\phi_n(x)| \le |\phi_n(x+y)| + |\phi_n(x)| \le |f(x+y)| + |f(x)|$, which is L^1,

hence for $y\to\infty$ and apply DCT.

$$2\int|f| = \lim_{n\to\infty} 2\int|\phi_n| = \lim_{n\to\infty} \int |\phi_n(x+y)-\phi_n(x)|$$

$$= \int \lim_{n\to\infty} |\phi_n(x+y)-\phi_n(x)|$$

$$= \int |f(x+y)-f(x)|$$
