REAL ANALYSIS QUALIFYING EXAM
SPRING 2004

Answer all four questions. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Show that

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+\frac{1}{n}) - \cos(x)}{x^{3/2}}\,dx$$

exists.

2. Suppose f_n, $n = 1, 2, \ldots$, and f are non-negative measurable functions on a measure space $(X, \mathcal{M}, \mu)$ with $f_n \to f$ a.e. and

$$\int_X f_n(x)\,d\mu(x) \to \int_X f(x)\,d\mu(x).$$

Show that

$$\int_X f_n(x)g(x)\,d\mu(x) \to \int_X f(x)g(x)\,d\mu(x)$$

for every bounded measurable function g. [Hint: use Fatou's Lemma].

3. Let $f \in L^1(\mathbf{R^d})$. Evaluate

$$\lim_{y\to\infty} \int_{\mathbf{R^d}} |f(x+y) - f(x)|\,dx.$$

[Note that the limit is NOT as $y \to 0$.]

4. Let E_n be the set of all $f \in C([0,1])$ for which there exists $x_0 \in [0,1]$ (depending on f) such that

$$|f(x) - f(x_0)| \le n|x - x_0| \quad \text{for all } x \in [0,1].$$

(i) Show that E_n is nowhere dense in $C([0,1])$.
(ii) Show that the set of nowhere differentiable functions $f \in C([0,1])$ is non-empty.