525 a

Real Analysis Qualifying Exam
USC Department of Mathematics
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Instructions. Do *four* out of the *five* problems, *on separate pieces of paper*. Be sure to justify your work.

Problem 1. $\{f_n\}$ is a sequence of measurable real-valued functions on a measure space $(X, \mathcal{A}, \mu)$.

(i) Suppose that $f_n \to f$ in measure and $|f_n| \leq g \in L^1(d\mu)$. Show that $f_n \to f$ in $L^1(d\mu)$.
(ii) Show that the result in (i) is false if the condition $|f_n| \leq g \in L^1(d\mu)$ is omitted.

Problem 2. For $a > 0$, show that

$$\int_0^\infty e^{-ax}x^{-1}\sin x\,dx = \arctan(a^{-1})$$

by integrating $e^{-axy}\sin x$ with respect to x and y.

Problem 3. Let A and B be Borel measurable subsets of a circle C of circumference 1 centered at the origin. Let A_t denote the set A rotated about the origin through an arc of length t. Prove that there exists a value of t such that

$$m(A_t \cap B) \geq m(A)m(B),$$

where m denotes the arclength measure.

Problem 4. Consider the functions $f_n(x) = n^\alpha x e^{-nx^2}$ to be integrated with respect to Lebesgue measure over the interval $E = [0, 1]$.

(i) Determine the values of the constant α for which the dominated convergence theorem applies.
(ii) Determine the values of α for which

$$\lim_{n\to\infty}\int_E f_n = \int_E \lim_{n\to\infty} f_n$$

Problem 5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *Lipschitzian* if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in \mathbb{R}$. Prove: if f is Lipschitzian and $\Omega \subset \mathbb{R}$ has Lebesgue measure zero, then the image $f(\Omega)$ has Lebesgue measure zero.