ANALYSIS QUALIFYING EXAM

MAY 10, 2001

Problem	Score
1	
2	
3	
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6	
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8	
Total	

Last Name: ____________________

First Name: ____________________

Social Security Number: ____________________

If you are taking the Real Analysis (525a) exam only: Do problems 1–4.

If you are taking the Complex Analysis (520) exam only: Do problems 5–8.

If you are taking both parts: In order to pass, you must do well on both the Real and Complex Analysis parts—high performance on one portion does not compensate for low performance on the other.

Do as many problems as you can.

REAL ANALYSIS PROBLEMS

(1) Suppose f and g are absolutely continuous functions on an interval (a, b).
   (a) Show that f is bounded on (a, b).
   (b) Show that fg is absolutely continuous on (a, b).

(2) Suppose $(X, \mathcal{M}, \mu)$ is a measure space with $\mu(X) < \infty$, and $\{f_n\}$ is a sequence of finite measurable functions with $f_n \to 0$ in measure.

(a) Show that

$$\int_X \frac{|f_n|}{1+|f_n|}\, d\mu \to 0.$$

(b) Give an example to show (a) is false if we remove the assumption $\mu(X) < \infty$.

(3)(a) Suppose ν is a finite measure on $(X, \mathcal{M})$ with $\nu(X) > 0$, and ν is *atomless*, that is, every $E \in \mathcal{M}$ with $\nu(E) > 0$ has a subset $F \in \mathcal{M}$ with $0 < \nu(F) < \nu(E)$. Show that for every $\epsilon > 0$ there exists $A \in \mathcal{M}$ with $0 < \nu(A) < \epsilon$.

(b) Suppose μ is another finite measure on $(X, \mathcal{M})$, and $\alpha = \sup\left\{\frac{\mu(E)}{\nu(E)} : E \in \mathcal{M}, \nu(E) > 0\right\} < \infty$. Show that μ is absolutely continuous with respect to ν. HINT: Use (a).

(c) For $A, B \in \mathcal{M}$ we say A is a *ν-essential subset* of B if there exists $N \in \mathcal{M}$ with $\nu(N) = 0$, $A \backslash N \subset B$. Suppose there exists at least one set which achieves the supremum in (b). Show that there exists a measurable set Y such that $\mu(F) = \alpha\nu(F)$ if and only if F is a ν-essential subset of Y. HINT: Use (b).

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable and let $g : [0, 1] \to \mathbb{R}$ be Lebesgue integrable. Show that the function

$$h(x) = \int_0^1 f(x - y)g(y)\, dy$$

is infinitely differentiable.