**ANALYSIS QUALIFYING EXAM** **SPRING 2000**

In order to pass, you must do well on both the Real and Complex Analysis parts - high performance on one portion does not compensate for low performance on the other.

Start each problem on a fresh sheet of paper, and write on only one side of the paper.

**REAL ANALYSIS.** Answer any three of the four questions.

1. Suppose that $F_n$, $n \geq 1$ are nondecreasing right-continuous functions on $\mathbf{R}$ and $F = \sum F_n$ is finite.

(a) Show that $F$ is right continuous.

(b) Suppose $F_n' = 0$ a.e. for all $n$. Show that $F' = 0$ a.e.

[Hint: consider the corresponding measures $\mu_n$, $\mu$ with $\mu_n((a,b]) = F_n(b) - F_n(a)$].

2. Let $(X, \mathcal{B}, \nu)$ be a finite measure space and $f_n$, $f$ non-negative bounded measurable functions on $X$. Define measures $\mu_n$, $\mu$ on $(X, \mathcal{B})$ by

$$\mu_n(A) = \int_A f_n \, d\nu \qquad \text{and} \qquad \mu(A) = \int_A f \, d\nu.$$

(a) Show that if $f_n \to f$ in $L^1(\nu)$ then $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$.

(b) Conversely show that if $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$ then $f_n \to f$ in $L^1(\nu)$.

3. Let $\mu^*$ be an outer measure on $X$ and suppose $Y$ is a $\mu^*$-measurable subset of $X$. Let $\nu^*$ be the restriction of $\mu^*$ to subsets of $Y$. Show that a set $E \subset Y$ is $\nu^*$-measurable if and only if $E$ is $\mu^*$-measurable.

4. Let $f, f_1, \ldots, f_n, \ldots$ be measurable functions from the measure space $(E, \mathcal{E}, \mu)$ to an open subset $\Omega$ of $\mathbf{R}^d$ such that for all $\varepsilon > 0$,

$$\mu\left(\{x \in E : \|f(x) - f_n(x)\| \geq \varepsilon\}\right) \to 0 \quad \text{as } n \to \infty.$$

Assume that $\mu$ is finite. Show that for all $\varepsilon > 0$ there exists a compact set $K \subset \Omega$ such that $\mu\left(\{x : f(x) \notin K\}\right) \leq \varepsilon$ and $\mu\left(\{x : f_n(x) \notin K\}\right) \leq \varepsilon$ for every $n$.

**COMPLEX ANALYSIS.** See next page.