ANALYSIS QUALIFYING EXAM **FALL 1999**

In order to pass, you must do well on both the Real and Complex Analysis parts - high performance on one portion does not compensate for low performance on the other.

Start each problem on a fresh sheet of paper, and write on only one side of the paper.

REAL ANALYSIS. Answer question 1 and any two of the other three questions.

1. Let $\{f_n\}$ be a sequence of functions on $(X, \mathcal{A}, \mu)$. Suppose $\{f_n\}$ is Cauchy in measure, that is, for every $\varepsilon > 0$ there exists N such that $m, n \geq N$ implies

$$\mu(\{x \in X : |f_n(x) - f_m(x)| > \varepsilon\}) < \varepsilon.$$

Show that there exists f such that $f_n \to f$ in measure. HINT: For $n_1 < n_2 < \dots$ you can write $f_{n_k} - f_{n_j} = \sum_{i=j+1}^{k}(f_{n_i} - f_{n_{i-1}})$.

2. Suppose $f : [0,1] \to \mathbf{R}$ is a nonnegative Lebesgue measurable function satisfying

$$\int f^n \, dm = \int f \, dm \quad \text{for all } n \geq 1.$$

Show that f is the characteristic function χ_E of some measurable $E \subset [0,1]$.

3. Let $(X, \mathcal{A}, \mu)$ be a measure space and let m denote Lebesgue measure on $[0,1]$. Suppose F_n and F map $X \times [0,1]$ into $\mathbf{R}$ and satisfy

(i) $F_n(x, \cdot)$ is absolutely continuous and nondecreasing for all n, x;

(ii) $F(x, \cdot)$ is continuously differentiable for all x;

(iii) $\dfrac{\partial}{\partial t}F_n(x,t) \to \dfrac{\partial}{\partial t}F(x,t)$ for almost every x, t;

(iv) $|F_n(x,t) - F_n(x,0)| \leq t g(x)$ for all n, x, t, where g is an integrable function on X.

Show that

$$\frac{\partial}{\partial t}\int_X F(x,t)\, d\mu(x)\bigg|_{t=0} = \int_X \frac{\partial F}{\partial t}(x,0)\, d\mu(x).$$

4. Let X be a metric space and μ a regular Borel measure on $(X, \mathcal{B})$ with $\mu(X) = 1$. Let $\mathcal{E} = \{F \in \mathcal{B} : F \text{ closed}, \mu(F) = 1\}$ and $H = \bigcap_{F \in \mathcal{E}} F$. (Note *regular* means $\mu(E) = \sup\{\mu(K) : K \text{ compact}, K \subset E\} = \inf\{\mu(U) : U \text{ open}, E \subset U\}$ for all $E \in \mathcal{B}$.)

(a) Show that $\mathcal{E}$ is closed under finite intersections.

(b) Show that $\mu(H) = 1$. HINT: Show that $\mu(H^c) = 0$.

COMPLEX ANALYSIS. See next page.