Analysis Qualifying Exam, Fall 1997

	Problem	Points	Selection	Score
Last Name:	1	20	Y N	
First Name:	2	20	Y N	
Social Security Number:	3	20	Y N	
	4	20	Y N	
Please mark the 5 problems to be graded	5	20	Y N	
in the table on the right.	6	20	Y N	
	Total	100		

Problem 1. Let μ be a finite measure on $(X, \mathcal{B})$, and let α be a positive function on $\mathbb{R}$ such that $\alpha(x)/x \to \infty$ as $x \to \infty$. Suppose $f_n \to f$ a.e. on X, and $\|\alpha \circ f_n\|_1 \leq M < \infty$ for all $n \in \mathbb{N}$.

(a) Show that

$$\sup_n \|f_n \chi_{[f_n \geq K]}\|_1 \to 0, \quad \text{as} \quad K \to \infty.$$

(Here χ_A denotes the characteristic function of the set A.)

(b) Show that for each $K > 0$, $f_n \chi_{[f_n < K]} \to f \chi_{[f < K]}$ in L^1 as $n \to \infty$.

(c) Show that $f_n \to f$ in L^1 as $n \to \infty$.

Problem 2. Consider a nonnegative valued function $f \in L^1(\mu)$. Suppose $c \geq 0$, $A = \{x \in X : f(x) \geq c\}$, and $\mu(A) = \delta > 0$. Show that

$$\sup_{\{B:\mu(B)\leq\delta\}} \int_B f d\mu = \int_A f d\mu.$$

Problem 3. Show that for every $f, g \in L^1(\mathbb{R}, \mu)$, the following equality holds

$$\lim_{h \downarrow 0} \frac{1}{h}\left(\int |f + hg| d\mu - \int |f| d\mu\right) = \int_{\Sigma_0^c} g(x)\mathrm{sign}(f(x)) d\mu + \int_{\Sigma_0} |g(x)| d\mu,$$

where μ is a measure on $\mathbb{R}$ and $\Sigma_0 = \{x : f(x) = 0\}$.

PROBLEM 4. Suppose the measures μ_n, $n \geq 1$, on $(X, \mathcal{B})$ are uniformly absolutely continuous with respect to some finite measure ν, that is, given $\epsilon > 0$ there is $\delta > 0$ such that $\nu(A) < \delta$ implies $\mu_n(A) < \epsilon$ for all n. Suppose also that $d\mu_n/d\nu \longrightarrow f$, ν-a.e. Show that there exists a measure μ such that $\mu_n(A) \longrightarrow \mu(A)$ for all measurable A. Identify the measure μ.

PROBLEM 5. Let μ be a finite measure on the Borel sets in a separable metric space X, and define $\text{supp}(\mu) = \{x \in X : \mu(D) > 0 \text{ for every open set D containing } x\}$ and $G = (\text{supp}(\mu))^c$.

(a) Show that G is open.

(b) Show that $\mu(G) = 0$.

(c) If B is open and $\mu(B) = 0$, show that $B \subset G$.

PROBLEM 6. Consider the following modes of convergence of functions $f_n \longrightarrow f$:

(i) almost everywhere
(ii) in measure
(iii) in L^1
(iv) almost uniformly
(v) uniformly.

(a) Which implications among these are valid in all measure spaces? You need not prove these; just list all of them or make a diagram.

(b) Let μ be counting measure on the positive integers. What additional implications, if any, are valid in this special case? (Prove these implications; you may use any of the implications in (a) without proof.)