REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1996

Problem 1 (Stability of contractive iteration) Let (M, d) be a metric space, and suppose $T : M \to M$ satisfies

$$d(Tx, Ty) \leq k \cdot d(Tx, Ty) \quad \text{for all } \ x, y \in M$$

where $0 < k < 1$. Now suppose $\varepsilon > 0$, and a sequence $\{\hat{x}_n\}_{n=0}^{\infty}$ in M satisfies

$$d(\hat{x}_n, T\hat{x}_{n-1}) < \varepsilon \quad \text{for all } \ n \geq 1$$

Prove that for $0 \leq m < n$,

$$d(\hat{x}_m, \hat{x}_n) < k^n \frac{2d(\hat{x}_0, T\hat{x}_0)}{1-k} + \frac{2\varepsilon}{1-k}$$

Problem 2 How many zeros does the polynomial $p(z) = z^4 - 2z + 3$ have in the unit disk $|z| < 1$?

Problem 3 Suppose $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable and

$$\int_{-\infty}^{\infty} \varphi(x) f(x) \, \mathrm{d}x = 0$$

for all continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ which have compact support. Prove: $f(x) = 0$ for a.e. x.

Problem 4 Evaluate

$$\int_0^{\pi} \frac{\mathrm{d}\theta}{2 + \sin\theta}$$

Problem 5 Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$, and let M denote the space of Σ-measurable extended-real-valued functions on X. Define $\rho : M \times M \to \mathbb{R}$ by

$$\rho(f, g) = \int \frac{|f-g|}{1 + |f-g|} \, \mathrm{d}\mu$$

Show that ρ is a metric on M, and that $f_n \to f$ in the ρ-metric iff $f_n \to f$ in measure.

Problem 6 Suppose $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Prove that there exists a point $z_0 \in \mathbb{C}$ such that we can expand $f(z)$ into a power series about z_0,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

for which all $c_n \neq 0$.

Problem 7 Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives f_{xy} and f_{yx}. Prove $f_{xy} \equiv f_{yx}$. **Hint:** use Fubini's theorem to integrate f_{xy} and f_{yx} over a rectangle $[a, b] \times [c, d]$.

Problem 8 Find a conformal mapping from the unit disk $|z| < 1$ to the region

$$\Omega = \{x + iy : (x < 0) \text{ and } (y > 0), \text{ or } (x \geq 0) \text{ and } (y > b)\}$$

where $b > 0$.