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Graduate Exam in Analysis, Fall 1995

1. We say that "$f_n \to f$ almost in $L^1(\mu)$" if for all $\epsilon > 0$ there exists a set N such that $\mu(N) < \epsilon$ and $\int_{N^c} |f_n - f| d\mu \to 0$ as $n \to \infty$.

a) Show that if f_n converges to f almost in L^1, and f_n converges to g almost in L^1, then $f = g$ a.e.

Consider the following statements:

(1) f_n converges to f in L^1.

(2) f_n converges to f almost in L^1.

(3) f_n converges to f in measure.

b) Show that (1) implies (2) implies (3).

c) Show that neither of the two reverse implications in part b) hold.

2. Let μ be a measure on the Borel subsets of R^n. With τ denoting the collection of open subsets of R^n, we define the support of μ by

$$support(\mu) = \{x \in R^n : \mu(U) > 0 \text{ for all } U \in \tau \text{ with } x \in U\}.$$

Prove that the set $support(\mu)$ is closed.

3. If f_n is a sequence of continuous functions on $[0, 1]$ with $f_n \to f$ a.e. m (Lebesgue measure), prove that for any $0 \leq a < 1$, $[0, 1]$ contains a compact subset K such that $m(K) > a$ and f is continuous on K. (Hint: Apply Egoroff's theorem.)

X 4. Let I be the collection of bounded open intervals (a, b) of R and m Lebesgue measure. Prove there is no Borel set E such that $m(A \cap E) = \frac{1}{2}m(A)$ for all $A \in I$.

5. Evaluate the following integral: $\int_0^\infty \frac{x \cos x}{x^2+1} dx$.

6. Conformally map the region $\{z = x + iy \in C : y > \frac{1}{4} - x^2\}$ to the unit disk $|z| < 1$.

7. Let $f(z)$ be a bounded analytic function on $|z| < 1$, and $\{z_n\}$ be the zeroes of $f(z)$, is it true that $\sum(1 - |z_n|) < \infty$?