# REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1994

**Problem 1** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$, must the derivative $f'$ be continuous at SOME point of $\mathbb{R}$? Explain. (Write $f'$ as the limit of a sequence of continuous functions.)

**Problem 2** Evaluate

$$\int_{f(\Gamma_\rho)} \frac{\mathrm{d}z}{1+z}$$

where $\Gamma_\rho$ is the square with verticies $\pm\rho$, $\pm i\rho$ and $f(z) = e^z$.

**Problem 3** Let $(X, \mathcal{B}, v)$ be a finite measure space and $f_n$, $f$ nonnegative bounded measurable functions on $X$. Define measures $\mu_n$, $\mu$ by

$$\mu_n(A) = \int_A f_n \, \mathrm{d}v, \quad \mu(A) = \int_A f \, \mathrm{d}v$$

Prove: $f_n \to f$ in $L^1$ iff $\sup_{A \in \mathcal{B}} |\mu_n(A) - \mu(A)| \to 0$ as $n \to \infty$.

**Problem 4** Evaluate $\int_0^\infty \frac{x^{1/3}}{4+x^4} \, \mathrm{d}x$

**Problem 5** Suppose $\{f_n\}$ is a sequence of continuously differentiable functions on $[0,1]$ which converges in the $L^2$ sense to 0, and whose derivatives $\{f_n'\}$ also converge to 0 in the $L^2$ sense. Prove: $\{f_n\}$ converges to 0 uniformly.

**Problem 6** Conformally map the open unit disk to a semi-infinite strip in the plane, $\{z \in \mathbb{C} : \Re z > 0, 0 < \Im z < a\}$ for some $a > 0$.

**Problem 7** Suppose $(X, \mathcal{B}, \mu)$ is a finite measure space and $f_n$, $g_n$ are measurable real-valued functions on $X$ such that $f_n \to f$ in measure and $g_n \to g$ in measure.

(a) Show that given $\varepsilon > 0$ there exists an $M$ such that $\mu(\{x \in X : |f_n(x)| > M\}) < \varepsilon$ for all $n$.

(b) Show that $f_n g_n \to fg$ in measure.