Fall 1993

Real Analysis Oval

② D countable dense subset of $(0,1)$ & G open subset of $\mathbb{R}$ ∋ $D \subseteq G$.

Then let $D = \{d_i\}$. Then for each d_i $\exists\ \delta_i > 0$ ∋ $B_{\delta_i}(d_i) \subseteq G$.

so $D \subseteq \bigcup B_{\delta_i}(d_i) \subseteq G$.

Now if $(0,1) \not\subseteq \bigcup B_{\delta_i}(d_i)$ $\exists\ d_i, d_{i+1}$ ∋ $B_{\delta_i}(d_i) \cap B_{\delta_{i+1}}(d_{i+1}) = \emptyset$

$\Rightarrow F = (d_i + \delta_i\ ,\ d_{i+1} - \delta_{i+1}) \cap (\bigcup B_{\delta_i}(d_i)) = \emptyset$

But $d_i, d_{i+1} \in D \subseteq (0,1)$ so $(d_i + \delta_i\ ,\ d_{i+1} - \delta_{i+1}) \subseteq (0,1)$

Hence for any $x \in F$ $\exists\ \epsilon > 0$ ∋ $B_\epsilon(x) \subseteq F \subseteq (0,1)$.

But then $B_\epsilon(x) \cap (\bigcup B_{\delta_i}(d_i)) = \emptyset \Rightarrow B_\epsilon(x) \cap D = \emptyset$ $\Rightarrow \Leftarrow$

since D is dense in $(0,1)$.

Thus $(0,1) \subseteq \bigcup B_{\delta_i}(d_i) \subseteq G$. ◻

⑤ $f : (a,b) \to \mathbb{R}$ is absolutely continuous $\Rightarrow$ f is uniformly conts.
$f \in BV$.

<u>uniformly continuous</u>: We know $\forall \epsilon > 0\ \exists \delta$ ∋ for any disjoint $\bigsqcup^n (a_i, b_i)$
∋ $\sum^n b_i - a_i < \delta$ $\Rightarrow$ $\sum^n |F(b_i) - F(a_i)| < \epsilon$.

Hence given any $x, y \in (a,b)$ & $n=1$ we have that $|x-y| < \delta$
$\Rightarrow |F(x) - F(y)| < \epsilon$ $\forall x, y \in (a,b)$. so F is uniformly conts.

<u>BV</u>: We want $T_F(b) - T_F(a) = \sup \{ \sum_{i=1}^n |F(x_i) - F(x_{i-1})| \ a = x_1 \leq \dots < x_n = b \}$
to be finite.

Now given any partition $\{x_i\}_{i=1}^n$ we can refine it into partitions of length at most δ.

Hence let $\{y_{ij}\}_{j=1}^{m_i}$ be a subdivision of $[x_i, x_{i+1}]$. $x_i = y_{i1}$
$x_{i+1} = y_{im_i}$

Then $|F(x_{i+1}) - F(x_i)| = |\sum_{j=1}^{m_i} F(y_{i+1}) - F(y_i)| \le \sum^{m_i} |F(y_{i+1}) - F(y_i)| < m_i \cdot \epsilon$

since by construction $|y_i - y_{i-1}| < \delta$.

So then $\sum^{n} |F(x_i) - F(x_{i-1})| \le n \cdot m_i \cdot \epsilon < M \cdot \epsilon < \infty$.

Since ϵ is arbitrary & thus holds for all partitions we obtain

$T_F(b) - T_F(a) < \infty$ as desired.

⑥ Show $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ & find its L^2 norm.

want $\int_0^\infty \left|\frac{\sin x}{x}\right|^2 dx < \infty$. $\Leftrightarrow$ $\int_0^\infty \frac{\sin^2 x}{x^2} dx < \infty$

$u = \sin^2 x$ $\quad v' = \frac{1}{x^2}$

$\Rightarrow \int \frac{\sin^2 x}{x^2} dx = -\frac{\sin^2 x}{x} + \int \frac{2 \sin x \cos x \, dx}{x}$ $\quad u' = 2 \sin \cos x$ $\quad v = -\frac{1}{x}$

$= -\frac{\sin^2 x}{x} + \int \frac{\sin 2x}{x} dx$

since $\sin 2x = 2 \sin x \cos x$
by $i \sin n\theta + \cos n\theta = (i \sin \theta + \cos \theta)^n$

$= -\frac{\sin^2 x}{x} + 4 \int \frac{\sin x}{x} dx$

Now $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$ (by use of Residue's & $\int \frac{e^{iz}}{z} dz$)

also $\lim_{x \to \infty} \frac{\sin^2 x}{x} \le \lim_{x \to \infty} \frac{1}{x} = 0$ & $\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} 2 \sin x \cos x = 0$

so $\int_0^\infty \frac{\sin^2 x}{x^2} dx = 4\left(\frac{\pi}{2}\right) = 2\pi$

hence $\left\| \frac{\sin x}{x} \right\|_2 = \sqrt{2\pi}$.

D

⑦ f-nonnegative $(f \geq 0)$ Lebesgue integrable on $[0,1]$

$\left(\int_0^1 |f| dx < \infty \right)$ & $\{r_n\}_{n=1}^{\infty} = \mathbb{Q}$

<u>WANT</u> $\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|) < \infty$ a.e. $x \in [0,1]$.

Consider: $\int_0^1 \sum_{n=1}^{\infty} \frac{1}{2^n} f(|x-r_n|) dx = \int_0^1 \lim_{n \to \infty} \left(\sum_{i=1}^{n} \frac{1}{2^i} f|x - r_i| \right) dx$

Since $\int_0^1 |f| dx < \infty \Rightarrow |f(x)| < \infty$ a.e. $x \in [0,1]$.

Hence $|f(x)| < M$ a.e. $x \in [0,1]$ and some $M \in \mathbb{R}$.

so $\left| \sum^{n} \frac{1}{2^i} f(|x - r_i|) \right| \leq \sum^{n} \frac{1}{2^i} \cdot M \leq M \cdot \sum^{\infty} \frac{1}{2^i} = 2M \ \forall n$.

now since $\sum^{n} \frac{1}{2^i} f(|x - r_i|) \to \sum^{\infty} \frac{1}{2^i} f(|x-r_i|)$ & $\int_0^1 2M dx < \infty$

Then by LDCT (Note $\sum^{n} \frac{1}{2^i} f(|x-r_i|) \in L^1([0,1])$ by (*) below):

$$\int_0^1 \lim_{n \to \infty} \sum^{n} \frac{1}{2^i} f(|x - r_i|) dx = \sum_{i=1}^{\infty} \left[\frac{1}{2^i} \int_0^1 f(|x - r_i|) dx \right]$$

$$\text{Now } \int_0^1 f(|x-r_i|) dx = \int_{-r_i}^{1-r_i} f(|t|) dt = \int_{-r_i}^{0} f(|t|) dt + \int_0^{1-r_i} f(|t|) dt$$

$$= \int_0^{r_i} f(t) dt + \int_0^{1-r_i} f(t) dt < 2 \int_0^1 |f| dt < \infty$$

Hence (*) $\int_0^1 f(|x - r_i|) dx < N$ for some $N \in \mathbb{R}$ & $\forall i$.

$$\Rightarrow \int_0^1 \sum^{\infty} \frac{1}{2^i} f(|x - r_i|) dx \leq N \cdot \sum^{\infty} \frac{1}{2^i} = 2N < \infty$$

So then $\sum^{\infty} \frac{1}{2^i} f(|x - r_i|) < \infty$ a.e. $x \in [0,1]$.

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