REAL AND COMPLEX ANALYSIS QUALIFYING EXAM

FALL 1993

Problem 1 Define $D_r = \{z \in \mathbb{C} : |z| < r\}$, the open r-disk. Let $M > 0$ and $f_n : D_1 \to D_M$ for $n = 1, 2, \ldots$ be a sequence of analytic functions. Prove there is a subsequence which converges uniformly on $D_{1/2}$.

Problem 2 Prove or find a counterexample: Let D be a countable dense subset of $(0, 1)$ and let G be an open subset of $\mathbb{R}$ such that $G \supset D$, then $G \supset (0, 1)$.

Problem 3 Let f be a non-constant meromorphic function which is doubly periodic (i.e. has two periods linearly independent over the reals). Prove that f has at least one singularity.

Problem 4 How many roots of the equation $f(z) = 0$ lie in the right half-plane, where

$$f(z) = z^4 + \sqrt{2}z^3 + 2z^2 - 5z + 2$$

Hint: consider the image of the imaginary axis.

Problem 5 Show that a function $f : (a, b) \to \mathbb{R}$ which is absolutely continuous is both uniformly continuous and of bounded variation.

Problem 6 Show that $\frac{\sin x}{x} \in L^2(\mathbb{R}^+)$ and evaluate its L^2 norm.

Problem 7 Suppose f is a non-negative function which is Lebesgue integrable on $[0, 1]$, and $\{r_n : n = 1, 2, \ldots\}$ is an enumeration of the rational numbers in $[0, 1]$. Show that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} f(|x - r_n|)$$

converges for a.e. $x \in [0, 1]$.