Qualifying Exam: Real Analysis

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1. Let $(X, \mathscr{M}, \mu)$ be a measure space. Assume that μ is semifinite: for every $E \in \mathscr{M}$ such that $\mu(E) > 0$, there exists $A \subseteq E$ such that $A \in \mathscr{M}$ and $0 < \mu(A) < \infty$. Prove that for every $E \in \mathscr{M}$,

$$\mu(E) = \sup\{\mu(A) : A \subseteq E, A \in \mathscr{M}, \mu(A) < \infty\}.$$

Solution. Let $\mathscr{A}_E = \{A \subseteq E, A \in \mathscr{M}, \mu(A) < \infty\}$. By monotonicity, it suffices to prove $\mu(E) \leq \sup\{\mu(A) : A \in \mathscr{A}_E\}$. If $\mu(E) < \infty$, then $E \in \mathscr{A}_E$ and we are done; thus, suppose that $\mu(E) = \infty$. Assume for the sake of contradiction that $\mu(E) > \sup\{\mu(A) : A \in \mathscr{A}_E\} = s$. The supremum is attained by some $A^\star \in \mathscr{A}_E$ (e.g., $\bigcup_{n=1}^\infty A_n$ where $A_n \in \mathscr{A}_E$ and $\mu(A_n) > s - \frac{1}{n}$ for all n). But the semifiniteness of μ yields some $B \in \mathscr{A}_{E \setminus A^\star}$ with $\mu(B) > 0$, which leads to

$$\begin{cases} A^\star \sqcup B \in \mathscr{A}_E, \\ \mu(A^\star \sqcup B) = \mu(A^\star) + \mu(B) > \mu(A^\star) = s, \end{cases}$$

a contradiction. We conclude that $\mu(E) \leq \sup\{\mu(A) : A \in \mathscr{A}_E\}$.

2. Compute

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) \mathrm{d}x.$$

Solution. The dominated convergence theorem, with $(1 + \frac{x}{2})^{-2} \cdot 1$ as the dominating function, implies that

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) \mathrm{d}x = \int_0^\infty \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) \mathrm{d}x = \int_0^\infty e^{-x} \cdot 0 \, \mathrm{d}x = 0.$$

3. Let $f \in L^1([0,1], m)$, and let $A_\varepsilon = \{(x, y) \in [0,1]^2 : |x - y| < \varepsilon\}$. Prove that

$$\iint_{A_\varepsilon} |f(x) - f(y)| \,\mathrm{d}x \,\mathrm{d}y \le 4\varepsilon \|f\|_{L^1}.$$

Solution. The triangle inequality and Tonelli's theorem imply that

$$\iint_{A_\varepsilon} |f(x) - f(y)| \,\mathrm{d}x \,\mathrm{d}y \le 2 \iint_{A_\varepsilon} |f(x)| \,\mathrm{d}y \,\mathrm{d}x \le 2 \int_0^1 \int_{x-\varepsilon}^{x+\varepsilon} |f(x)| \,\mathrm{d}y \,\mathrm{d}x = 4\varepsilon \int_0^1 |f(x)| \,\mathrm{d}x.$$

Prove that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iint_{A_\varepsilon} |f(x) - f(y)| \,\mathrm{d}x \,\mathrm{d}y = 0.$$

Hint: Apply a change of variables.

Remark. Draw a picture of A_ε. Rotate it 45 degrees clockwise.

Solution. Let $s = x + y$ and $t = y - x$, and suppose without loss of generality that $f = f \cdot \mathbb{1}_{[0,1]}$. Then,

$$\iint_{A_\varepsilon} |f(x) - f(y)| \,\mathrm{d}x \,\mathrm{d}y = \frac{1}{2} \int_{[-\varepsilon,\varepsilon]} \int_{[0,2]} |f(\tfrac{s-t}{2}) - f(\tfrac{s+t}{2})| \,\mathrm{d}s \,\mathrm{d}t.$$

The inequality of part (a) implies that the function $t \mapsto \int_0^2 |f(\frac{s-t}{2}) - f(\frac{s+t}{2})| \,\mathrm{d}s$ belongs to $L^1(\mathbb{R})$. The Lebesgue differentiation theorem then implies that

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{[-\varepsilon,\varepsilon]} \int_{[0,2]} |f(\tfrac{s-t}{2}) - f(\tfrac{s+t}{2})| \,\mathrm{d}s \,\mathrm{d}t = \int_{[0,2]} |f(\tfrac{s}{2}) - f(\tfrac{s}{2})| \,\mathrm{d}s = 0.$$

4. Let n and p be integers such that $1 \le p < n$. Assume that $E_1, \ldots, E_n$ are measurable subsets of $[0,1]$ such that each point $x \in [0,1]$ belongs to at least p of the sets $E_1, \ldots, E_n$. Prove that there exists $j \in \{1, \ldots, n\}$ such that

$$m(E_j) \ge \frac{p}{n}.$$

Solution. We are given the inequality $\sum_{i=1}^n \mathbb{1}_{E_i}(x) \ge p$ for every $x \in [0,1]$. Integration yields

$$\sum_{i=1}^n m(E_i) = \int_{[0,1]} \sum_{i=1}^n \mathbb{1}_{E_i} \,\mathrm{d}m \ge \int_{[0,1]} p \,\mathrm{d}m = p.$$

If the sum of n nonnegative numbers is at least p, then one of the numbers is at least $\frac{p}{n}$.