Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu

Fall 2024

1. Compute

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{x^2-5x+1}{1+x^{6n}}\,\mathrm{d}x.$$

Solution. The dominated convergence theorem with dominating function

$$\sup_n\left|\frac{x^2-5x+1}{1+x^{6n}}\right|\le\begin{cases}(6x^2+1)x^{-6} & \text{if } |x|\ge 1,\\ 7 & \text{if } |x|<1\end{cases}$$

yields the answer

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{x^2-5x+1}{1+x^{6n}}\,\mathrm{d}x=\int_{-\infty}^{\infty}\lim_{n\to\infty}\frac{x^2-5x+1}{1+x^{6n}}\,\mathrm{d}x=\int_{-1}^{1}x^2-5x+1\,\mathrm{d}x=\frac{8}{3}.$$

2. Prove that if $f\colon[0,1]\to\mathbb{R}$ is nondecreasing and almost everywhere differentiable, then

$$\int_0^1 f'(x)\,\mathrm{d}x\le f(1)-f(0).$$

Solution. With the assumption that $f(x)=f(1)$ for all $x>1$, define

$$g_n(x)=\frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}.$$

Note the inequality

$$\int_0^1 g_n(x)\,\mathrm{d}x=n\int_{\frac{1}{n}}^{1+\frac{1}{n}}f(x)\,\mathrm{d}x-n\int_0^1 f(x)\,\mathrm{d}x=n\int_1^{1+\frac{1}{n}}f(x)\,\mathrm{d}x-n\int_0^{\frac{1}{n}}f(x)\,\mathrm{d}x\le f(1)-f(0).$$

Because $f'(x)$ is defined and equal to $\lim_{n\to\infty}g_n(x)$ almost everywhere, Fatou's lemma implies that

$$\int_0^1 f'(x)\,\mathrm{d}x=\int_0^1\liminf_{n\to\infty}g_n(x)\,\mathrm{d}x\le\liminf_{n\to\infty}\int_0^1 g_n(x)\,\mathrm{d}x\le f(1)-f(0).$$

3. Let E be a subset of $\mathbb{R}$ with finite Lebesgue measure, and let $f \in L^1(\mathbb{R})$. Prove that

$$\lim_{t\to\infty} \int_E f(x+t)\,\mathrm{d}x = 0.$$

Solution. Let $\varepsilon > 0$. The denseness of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$ yields $g \in C_c(\mathbb{R})$ such that $\|f - g\|_{L^1} < \frac{\varepsilon}{2}$ and hence

$$\int_E |f(x+t)|\,\mathrm{d}x \le \int_{\mathbb{R}} |f(x+t) - g(x+t)|\,\mathrm{d}x + \int_E |g(x+t)|\,\mathrm{d}x < \frac{\varepsilon}{2} + \int_E |g(x+t)|\,\mathrm{d}x.$$

Suppose that support$(g) \subseteq [-a, a]$ and $|g| \le M$. Continuity from below yields n such that $m(E \setminus [-n, n]) < \frac{\varepsilon}{2M}$. For all $t > a + n$, then,

$$\int_E |g(x+t)|\,\mathrm{d}x = \int_{E+t} |g(x)|\,\mathrm{d}x = \int_{(E\setminus[-n,n])+t} |g(x)|\,\mathrm{d}x \le m(E \setminus [-n, n]) \cdot M < \frac{\varepsilon}{2}.$$

Remark. Cf. Problem 2 on the Fall 2021 exam.

4. Let f and g be integrable functions on a complete measure space $(X, \mathscr{M}, \mu)$, let $F_t = \{x \in X : f(x) > t\}$, and let $G_t = \{x \in X : g(x) > t\}$. Prove that

$$\int_X |f - g|\,\mathrm{d}\mu = \int_{-\infty}^{\infty} \mu(F_t \triangle G_t)\,\mathrm{d}t.$$

Solution. For each n, $\{x : |f(x) - g(x)| > \frac{1}{n}\}$ has finite measure by Markov's inequality, so $E = \{x : |f(x) - g(x)| > 0\} = \bigcup_{n=1}^{\infty} \{x : |f(x) - g(x)| > \frac{1}{n}\}$ is σ-finite. Tonelli's theorem, and the fact that $\mathbb{1}_{F_t \triangle G_t}(x) = |\mathbb{1}_{F_t}(x) - \mathbb{1}_{G_t}(x)| = 1$ precisely when t is between $f(x)$ and $g(x)$, implies that

$$\int_{-\infty}^{\infty} \mu(F_t \triangle G_t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \int_E \mathbb{1}_{F_t \triangle G_t}(x)\,\mathrm{d}\mu\,\mathrm{d}t = \int_E \int_{-\infty}^{\infty} \mathbb{1}_{F_t \triangle G_t}(x)\,\mathrm{d}t\,\mathrm{d}\mu = \int_E |f(x) - g(x)|\,\mathrm{d}\mu = \int_X |f - g|\,\mathrm{d}\mu.$$

Remark. Cf. Problem 2 on the Spring 2024 exam.