REAL ANALYSIS GRADUATE EXAM

Fall 2024

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate the limit and justify the steps along the way:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{x^2 - 5x + 1}{1 + x^{6n}}\, dx.$$

2. Let f be a non-decreasing function on $[0, 1]$. Assume that f is differentiable a.e.. Prove that $\int_0^1 f'(x)\, dx \le f(1) - f(0)$.

3. Assume that $E \subset \mathbb{R}$ is of finite Lebesgue measure, and let $f \in \mathscr{L}^1(\mathbb{R})$. Prove that

$$\lim_{t\to\infty} \int_E f(x + t)\, dx = 0.$$

4. Let f, g be real valued integrable functions on a complete measure space $(X, \mathcal{B}, \mu)$ and define

$$F_t = \{x \in X : f(x) > t\}, \qquad G_t = \{x \in X : g(x) > t\}.$$

Prove that

$$\int_X |f - g|\, d\mu = \int_{-\infty}^{\infty} \mu((F_t \setminus G_t) \cup (G_t \setminus F_t))\, dt,$$

where $F_t \setminus G_t = \{x \in F_t : x \notin G_t\}$ and similarly $G_t \setminus F_t = \{x \in G_t : x \notin F_t\}$.