Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu

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1. Let $f\colon \mathbb{R} \to [0,\infty)$ be a function in $L^1(\mu)$. Prove that for every $\varepsilon > 0$, there exists $A \subseteq \mathbb{R}$ such that $\mu(A) < \infty$ and

$$\int_A f \,\mathrm{d}\mu \geq \int_{\mathbb{R}} f \,\mathrm{d}\mu - \varepsilon.$$

Solution. Let $E_n = \{x \in \mathbb{R} : f(x) \leq n\}$. The monotone convergence theorem yields N such that

$$\int_{E_N} f \,\mathrm{d}\mu \geq \int_{\mathbb{R}} f \,\mathrm{d}\mu - \frac{\varepsilon}{2}$$

and, for this value of N, yields M such that

$$\int_{E_N \setminus E_{1/M}} f \,\mathrm{d}\mu \geq \int_{E_N} f \,\mathrm{d}\mu - \frac{\varepsilon}{2}.$$

Therefore, let $A = E_N \setminus E_{1/M}$. Markov's inequality implies that

$$\mu(A) \leq \mu\big(\{x \in \mathbb{R} : f(x) > \tfrac{1}{M}\}\big) \leq M \int_{\mathbb{R}} f \,\mathrm{d}\mu < \infty.$$

2. Prove or disprove the existence of

$$\lim_{n\to\infty} \int_0^1 \frac{(3x-1)^{2n}}{1+(3x-1)^{2n}} \,\mathrm{d}x.$$

Solution. The convenient change of variables $y = 3x-1$ and the bounded convergence theorem yield

$$\lim_{n\to\infty} \int_0^1 \frac{(3x-1)^{2n}}{1+(3x-1)^{2n}} \,\mathrm{d}x = \lim_{n\to\infty} \frac{1}{3} \int_{-1}^2 \left(1 - \frac{1}{1+y^{2n}}\right) \mathrm{d}y = \frac{1}{3} \int_{-1}^2 \left(\frac{\mathbb{1}_{y\in\{\pm1\}}}{2} + \mathbb{1}_{y\in(1,2)}\right) \mathrm{d}y = \frac{1}{3}.$$

3. Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is such that $\int_{\mathbb{R}^n} |f|\,\mathrm{d}y > 0$, then the Hardy–Littlewood maximal function

$$\mathrm{M}f(x) = \sup_{r>0} \frac{1}{m(\mathrm{Ball}(x,r))} \int_{\mathrm{Ball}(x,r)} |f(y)|\,\mathrm{d}y$$

does not belong to $L^1(\mathbb{R}^n)$.

Solution. Because $\int_{\mathbb{R}^n} |f|\,\mathrm{d}y > 0$, the monotone convergence theorem yields $r > 0$ such that

$$0 < C \overset{\text{def}}{=} \int_{\mathrm{Ball}(0,r)} |f(y)|\,\mathrm{d}y < \infty.$$

For $|x| > r$, the inclusion $\mathrm{Ball}(x, |x| + r) \supseteq \mathrm{Ball}(0, r)$ implies the inequality

$$\mathrm{M}f(x) \geq \frac{1}{m(\mathrm{Ball}(x,|x|+r))} \int_{\mathrm{Ball}(x,|x|+r)} |f(y)|\,\mathrm{d}y \geq \frac{1}{m(\mathrm{Ball}(x,|x|+r))} C.$$

Let V be the volume of the unit ball. Then,

$$\int_{\mathbb{R}^n} \mathrm{M}f(x)\,\mathrm{d}x \geq \frac{C}{V} \int_{\{|x|>r\}} \frac{1}{(|x|+r)^n}\,\mathrm{d}x = \infty.$$

4. Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is Lebesgue-measurable, then there exists a Borel-measurable function $g\colon \mathbb{R} \to \mathbb{R}$ such that $f = g$ almost everywhere with respect to the Lebesgue measure.

Solution. Because the Lebesgue σ-algebra is the completion of the Borel σ-algebra, each Lebesgue subset E contains a Borel subset B such that $E \setminus B$ has Lebesgue measure zero, so $\mathbb{1}_E = \mathbb{1}_B$ almost everywhere. Taking linear combinations and pointwise limits yields the result for general Lebesgue-measurable f.