Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu

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1. Let $f\colon \mathbb{R}\to\mathbb{R}$ be a Lebesgue-measurable function, and suppose that E is a measurable subset of $\mathbb{R}$ such that $0<\int_E f(x)\,\mathrm{d}x<\infty$. Show that for every $t\in(0,1)$, there exists a measurable set $E_t\subseteq E$ for which

$$\int_{E_t} f(x)\,\mathrm{d}x = t\int_E f(x)\,\mathrm{d}x.$$

Solution. Let $E_u = E\cap(-\infty,u]$, and define $F\colon\mathbb{R}\to\mathbb{R}$ by

$$F(u)=\int_{E_u} f(x)\,\mathrm{d}x.$$

Note that $F(-\infty)=0$, $F(\infty)=\int_E f(x)\,\mathrm{d}x$, and F is continuous: for every $u\in\mathbb{R}$ and every $\varepsilon>0$, the dominated convergence theorem (with $|f|\cdot\mathbb{1}_E$ as the dominating function) yields N such that

$$|u-u'|<\frac{1}{N} \implies |F(u)-F(u')|\le\int_{E\cap(u-\frac{1}{N},u+\frac{1}{N})}|f(x)|\,\mathrm{d}x<\varepsilon.$$

Then, for every $t\in(0,1)$, the intermediate value theorem yields u_t such that

$$F(u_t)=\int_{E_{u_t}} f(x)\,\mathrm{d}x = t\int_E f(x)\,\mathrm{d}x.$$

2. Let $f\in L^1(\mathbb{R})$, and let $F(t)=\int_{\mathbb{R}} f(x)e^{itx}\,\mathrm{d}x$. Prove that $F\colon\mathbb{R}\to\mathbb{C}$ is continuous, and prove that

$$\lim_{t\to-\infty}F(t)=\lim_{t\to\infty}F(t)=0.$$

Note: There is a typo in the original problem statement. The codomain of F is $\mathbb{C}$, not $\mathbb{R}$.

Solution. Let $t\in\mathbb{R}$. For every sequence $(\delta_n)_{n\ge1}$ that converges to 0, by the dominated convergence theorem with $2|f|$ as the dominating function,

$$\lim_{n\to\infty}|F(t+\delta_n)-F(t)|\le\lim_{n\to\infty}\int_{\mathbb{R}}|f(x)|\cdot|e^{i\delta_n x}-1|\,\mathrm{d}x=0.$$

This shows that F is continuous at t.

For every $\varepsilon>0$, the denseness of step functions in $L^1(\mathbb{R})$ yields $g=\sum_{j=1}^n c_j\cdot\mathbb{1}_{[a_j,b_j]}$ such that

$$\lim_{t\to\infty}\int_{\mathbb{R}} f(x)e^{itx}\,\mathrm{d}x\le\lim_{t\to\infty}\int_{\mathbb{R}}|f(x)-g(x)|\,\mathrm{d}x+\lim_{t\to\infty}\sum_{j=1}^n|c_j|\left|\int_{a_j}^{b_j}e^{itx}\,\mathrm{d}x\right|<\varepsilon+\lim_{t\to\infty}\sum_{j=1}^n|c_j|\frac{2}{t}=\varepsilon.$$

This shows that $\lim_{t\to\infty}F(t)=0$. Symmetrically, $\lim_{t\to-\infty}F(t)=\lim_{t\to\infty}\int_{\mathbb{R}}-f(-x)e^{itx}\,\mathrm{d}x=0$. We remark that this result is the *Riemann–Lebesgue lemma*.

3. Compute

$$\lim_{n\to\infty}\int_0^n \left(1+\frac{x^2}{n}\right)^{-(n+1)} \mathrm{d}x.$$

Solution. The inequality

$$0 \le \int_n^\infty \frac{1}{(1+\frac{x^2}{n})^{n+1}}\,\mathrm{d}x \le \int_n^\infty \frac{1}{(1+x)^{n+1}}\,\mathrm{d}x = \frac{1}{n(n+1)^n}$$

and the dominated convergence theorem, with $(1+x^2)^{-2}$ as the dominating function, imply that

$$\lim_{n\to\infty}\int_0^n \frac{1}{(1+\frac{x^2}{n})^{n+1}}\,\mathrm{d}x = \lim_{n\to\infty}\int_0^\infty \frac{1}{(1+\frac{x^2}{n})^{n+1}}\,\mathrm{d}x = \int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

4. Let $f \in L^1(\mathbb{R})$, and consider the maximal function

$$\mathrm{M}f(x) = \sup_{h>0}\frac{1}{2h}\int_{x-h}^{x+h} |f(t)|\,\mathrm{d}t.$$

Prove that there exists a constant $A > 0$ such that for every $\alpha > 0$,

$$m(\{x\in\mathbb{R} : \mathrm{M}f(x) > \alpha\}) \le \frac{A}{\alpha}\|f\|_{L^1}.$$

Solution. This is the Hardy–Littlewood maximal inequality [Folland, Theorem 3.17].