Qualifying Exam: Real Analysis

Unofficial solutions by Alex Fu

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1. Let f be a Lebesgue-integrable function on $\mathbb{R}^d$. Prove that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$m(A) < \delta \implies \int_A |f| \, \mathrm{d}m < \varepsilon.$$

Solution. Let $E_n = \{x \in \mathbb{R}^d : |f(x)| > n\}$. The dominated convergence theorem yields N such that $\int_{E_N} |f| \, \mathrm{d}m < \varepsilon/2$. Then, with $\delta = \varepsilon/(2N)$,

$$m(A) < \delta \implies \int_A |f| \, \mathrm{d}m = \int_{A \cap E_N} |f| \, \mathrm{d}m + \int_{A \setminus E_N} |f| \, \mathrm{d}m < \frac{\varepsilon}{2} + m(A) \cdot N < \varepsilon.$$

Remark. What we have proven is that the measure $A \mapsto \int_A |f| \, \mathrm{d}m$ is absolutely continuous with respect to m.

2. Let f be a Lebesgue-integrable function on $\mathbb{R}$. Prove that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| \, \mathrm{d}x = 0.$$

Solution. Let $\varepsilon > 0$. The denseness of $C_c(\mathbb{R})$ in $L^1(\mathbb{R})$ yields $g \in C_c(\mathbb{R})$ such that $\|f - g\|_{L^1} < \varepsilon/2$. The continuity of g and the bounded convergence theorem imply that

$$\lim_{h \to 0} \int_{\mathbb{R}} |g(x+h) - g(x)| \, \mathrm{d}x = \int_{\mathbb{R}} \lim_{h \to 0} |g(x+h) - g(x)| \, \mathrm{d}x = 0.$$

Then, the triangle inequality implies that

$$\lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - f(x)| \, \mathrm{d}x \le \lim_{h \to 0} \int_{\mathbb{R}} |f(x+h) - g(x+h)| \, \mathrm{d}x + 0 + \int_{\mathbb{R}} |g(x) - f(x)| \, \mathrm{d}x < \varepsilon.$$

3. Let f and $f_1, f_2, \ldots$ be Lebesgue-measurable functions on $\mathbb{R}$. Prove that if there exists C such that for every n,

$$\int_{\mathbb{R}} |f_n(x) - f(x)| \, \mathrm{d}x \le \frac{C}{n^2},$$

then $f_n \to f$ almost everywhere as $n \to \infty$.

Solution. Let $g(x) = \limsup_{n \to \infty} |f_n(x) - f(x)|$. For each $k \ge 2$,

$$\int_{\mathbb{R}} g(x) \, \mathrm{d}x \le \int_{\mathbb{R}} \sup_{n \ge k} |f_n(x) - f(x)| \, \mathrm{d}x \le \int_{\mathbb{R}} \sum_{n \ge k} |f_n(x) - f(x)| \, \mathrm{d}x = \sum_{n \ge k} \int_{\mathbb{R}} |f_n(x) - f(x)| \, \mathrm{d}x \le \sum_{n=k}^{\infty} \frac{C}{n^2} \le \frac{C}{k-1}.$$

Then, $\int_{\mathbb{R}} g(x) \, \mathrm{d}x = 0$. Because g is nonnegative, $g = 0$ almost everywhere, i.e., $f_n \to f$ almost everywhere.

4. Let $f\colon [0,1] \to \mathbb{R}$ be a Lebesgue-measurable function that is positive almost everywhere. Prove that if $E_1, E_2, \dots$ are Lebesgue-measurable subsets of $[0,1]$ such that

$$\lim_{k\to\infty} \int_{E_k} f(x)\,\mathrm{d}x = 0,$$

then $\lim_{k\to\infty} m(E_k) = 0$.

Solution. Let $A_n = \{x : \frac{1}{n-1} > f(x) \geq \frac{1}{n}\}$, with $A_1 = \{x : f(x) \geq 1\}$. By Tonelli's theorem,

$$0 = \lim_{k\to\infty} \int f \cdot \mathbb{1}_{E_k}\,\mathrm{d}m \geq \lim_{k\to\infty} \int \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{1}_{A_n} \cdot \mathbb{1}_{E_k}\,\mathrm{d}m = \sum_{n=1}^{\infty} \frac{1}{n} \lim_{k\to\infty} \int \mathbb{1}_{A_n} \cdot \mathbb{1}_{E_k}\,\mathrm{d}m \geq 0.$$

A sum of nonnegative numbers is zero only if each of its terms is zero, so $\lim_{k\to\infty} m(A_n \cap E_k) = 0$ for all n. The equality $m(\bigsqcup_{n=1}^{\infty} A_n) = m(\{x : f(x) > 0\}) = 1$ and the dominated convergence theorem then imply that

$$\lim_{k\to\infty} m(E_k) = \lim_{k\to\infty} \sum_{n=1}^{\infty} m(A_n \cap E_k) = \sum_{n=1}^{\infty} \lim_{k\to\infty} m(A_n \cap E_k) = 0.$$

Remark. This is Problem 1 on the Spring 2023 exam.