$\boxed{\text{Fa20.1}}$

$f_n \geq 0$. f_n measurable on (X, μ).

$\int f_n \, d\mu = 1$ $\forall n \in \mathbb{N}$. Claim: $\limsup_n f_n(x)^{1/n} \leq 1$ μ-a.e.

$\limsup_n f_n(x)^{1/n} = \lim_{k \to \infty} \sup_{n \geq k} f_n(x)^{1/n}$

$\int \lim_{k \to \infty} \sup_{n \geq k} f_n(x)^{1/n} dx = \lim_{k \to \infty} \int \sup_{n \geq k} f_n(x)^{1/n} dx$

Assume not: assume $\limsup_n f_n(x)^{1/n} > 1$ on $E \subset X$ w/ $\mu(E) > 0$.

Then for each $x \in E$
$\exists$ subsequence n_k and $\varepsilon > 0$ s.t. $f_{n_k}(x)^{1/n_k} > 1 + \varepsilon$ $\forall n_k$.
$f_{n_k}(x) > (1+\varepsilon)^{n_k}$

$\lim_n \int f_n \, d\mu = 1 = \int \lim f_n \, d\mu$ not true nec.
take $\chi_{[n, n+1]}$ on $\mathbb{R}$. not dominated

$\int \lim f_n \, d\mu \leq 1$

$\int \liminf f_n \, d\mu \leq \liminf \int f_n \, d\mu = 1$

Root test: $\limsup_n |x_n|^{1/n} > 1 \Rightarrow \sum |x_n| = \infty$

So $\limsup_n f_n(x)^{1/n} > 1$ on E, $\mu(E) > 0$.

Then $\sum f_n(x) = \infty$ $\forall x \in E$

Then $\int \sum f_n(x) = \infty$

$\lim_{n \to \infty} n^{2/n}$

$g_n = \frac{f_n}{n^2}$ $\limsup_n g_n^{1/n} = \limsup_n \left(\frac{f_n}{n^2}\right)^{1/n}$

$= \limsup \frac{f_n^{1/n}}{n^{2/n}} = \limsup f_n^{1/n}$

$\Rightarrow \sum g_n(x) = \infty \Rightarrow \int \sum g_n(x) = \infty = \sum \int g_n(x) = \sum \frac{1}{n^2} < \infty$.

$\Rightarrow \Leftarrow$.

20Fa.1 Try 2

f_n measurable, ≥ 0. $\int f_n \, d\mu = 1 \ \forall n \in \mathbb{N}$.
Claim: $\limsup_n f_n(x)^{1/n} \leq 1$ μ-a.e.

Assume instead, for some $\varepsilon > 0$, $\limsup_n f_n(x)^{1/n} > 1 + \varepsilon \quad \forall x \in E$
where $\mu(E) > 0$.

$f_n > (1+\varepsilon)^n$

$\limsup_n f_n(x)^{1/n} = \lim_{k \to \infty} \sup_{n \geq k} f_n(x)^{1/n} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty}$

Let $E_n = \{x : f_n(x)^{1/n} > 1 + \varepsilon\}$

$E = \limsup_n E_n = \{x : \limsup_n f_n(x)^{1/n} > 1 + \varepsilon\}$?

$$\begin{aligned}
&\limsup_n E_n \\
&\quad = \bigcap_k \bigcup_{n \geq k} E_n \\
&\quad = \bigcap_k \{x : f_n(x)^{1/n} > 1+\varepsilon, \text{ for some } n \geq k\} \\
&\quad = \{x : f_n(x)^{1/n} > 1+\varepsilon, \text{ for some } n \geq k, \ \forall k\} \\
&\quad \Updownarrow \\
&\{x : \limsup_n f_n(x)^{1/n} > 1+\varepsilon\}. \checkmark
\end{aligned}$$

$$\begin{aligned}
m(E) &= m\left(\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} E_n\right) \\
&= \lim_{k \to \infty} m\left(\bigcup_{n \geq k} E_n\right) \quad * \\
&\leq \lim_{k \to \infty} \sum_{n \geq k} \frac{1}{(1+\varepsilon)^n} \\
&= 0.
\end{aligned}$$

ε arbitrary. $\Rightarrow$ result.

$\mu(E_n) \leq \frac{1}{(1+\varepsilon)^n} \quad \forall n.$

$\mu(E_n) \to 0.$

$$\begin{aligned}
1 = \int f_n \, d\mu &= \int_{E_n} f_n \, d\mu + \int_{E_n^c} f_n \, d\mu \\
&\geq \int_{E_n} (1+\varepsilon)^n \, d\mu + \int_{E_n^c} f_n \, d\mu \\
&= (1+\varepsilon)^n \mu(E_n) + \int_{E_n^c} f_n \, d\mu \\
&\geq (1+\varepsilon)^n \mu(E_n)
\end{aligned}$$

So $(1+\varepsilon)^n \mu(E_n) \leq 1 \quad \forall n$

$$\begin{aligned}
m\left(\bigcup_{n \geq k} E_n\right) &\leq \sum_{n \geq k} m(E_n) \\
&\leq \sum_{n \geq k} \frac{1}{(1+\varepsilon)^n} \\
&< \infty
\end{aligned}$$

since $1 + \varepsilon > 1$.

$\Rightarrow$ cont. from above

20 Fa. 2

$x = 0.n_1 n_2 n_3 \ldots \quad \in [0,1].$

$f(x) = \min_i (n_i)$. Claim: f measurable, constant a.e.

Define $f_K(x) = f_K(0.n_1 n_2 n_3 \ldots) = \min\{n_i\}_{i=1}^K$.

$\lim_{K\to\infty} f_K(x) = f(x) \quad \forall x.$

f_K measurable, since $f_K^{-1}(\{d\}) = \bigcup_{\{n_1,\ldots,n_K : \min\{n_i\}=d\}} [0.n_1 n_2 \cdots n_K 000\ldots, 0.n_1 n_2 \cdots (n_K+1) 000\ldots)$

= finite union of intervals.

f_K dominated by $g(x) = 9$, integrable.

DCT $\Leftarrow\Rightarrow$ f_K approaches a measurable function $\Rightarrow$ f measurable.

$f_1(x) = n_1$

$f_2(x) = \min\{n_1, n_2\}$

$\vdots$

$f(x) \stackrel{?}{=} \lim_{n\to\infty} f_n(x)$ ✓

$\exists\, n_i$ s.t. $f(x) = n_i$

Claim: $f = 0$ a.e.

$f_K^{-1}(\{0\}) = \bigcup [0.n_1 n_2 \cdots n_K 000\ldots, 0.n_1 n_2 \cdots (n_K+1) 000\ldots)$ measure $\frac{1}{10^K}$

in how many of $\{n_i\}_{i=1}^K$ is one of the n_i's $= 0$. $1 - \frac{9^K}{10^K} = 1 - \left(\frac{9}{10}\right)^K$

$\Rightarrow m(f_K^{-1}(\{0\})) = 1 - \left(\frac{9}{10}\right)^K \longrightarrow 1$ as $K \to \infty$.

20 Fa. 3

f cont. on $[0,1]$. $F(x) = \sup_{0 \le y \le x} f(y)$, $x \in [0,1]$.

(a) Is F continuous?

$\lim_{z \to x} F(z) = \lim_{z \to x} \sup_{0 \le y \le z} f(y)$

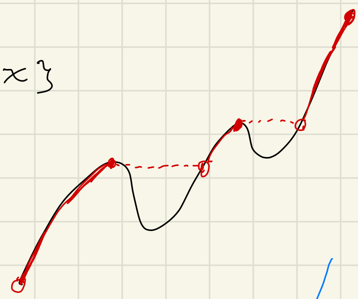
F is non-decreasing, since sup is being taken over a larger set as x increases.

$\lim_{z \nearrow x} F(z) = \lim_{z \nearrow x} \sup_{0 \le y \le z} f(y)$

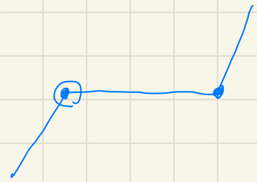
Since f is continuous on a compact set, the supremum is acheived: $\exists\ c \in [0,x]$ s.t. $F(x) = f(c)$. If $c \in [0,x)$, then $\exists\ N \in \mathbb{N}$ s.t. $\forall n \ge N$, $\sup_{0 \le y \le z_n} f(y) = f(c) = F(x)$ for $z_n \nearrow x$.
If $c = x$, then since f continuous, $\forall\ z_n \nearrow x$ $f(z_n) \nearrow f(x)$
$\Rightarrow \sup_{0 \le y \le z_n} f(y) \nearrow f(x) = F(x)$. Similarly F is continuous as $z \searrow x$.

Thus F continuous $\Rightarrow$ F Borel function.

(b) Define $S = \{x \in (0,1] : f(x) > f(y)\ \forall\ 0 \le y < x\}$



Since F is non-decreasing and continuous,
F' defined a.e. and
$S =$



Consider F^{-1} which is non decreasing and has countably many discontinuities,
$S = \bigcup F^{-1}$

continuous

$\{f - F = 0\}$

$S = \{f = F\} \setminus \bigcup \{F^{-1}(\{c\}) \mid m(F^{-1}(\{c\})) > 0\}$

$\bigcup_{c \in \mathbb{R}} \{F^{-1}(\{c\})\} = [0,1]$

20 Fa. 4

$f(x) = \sum_{n=1}^{\infty} a_n e^{-nx} \quad x \geq 0 \quad a_n > 0 \quad \forall n,$

$\sum_{n=1}^{\infty} a_n < \infty, \quad$ Claim: $\sum_{n=1}^{\infty} n a_n < \infty \iff f'(x+)$ exists at 0.

Note $\quad f'(x+) = \lim_{y \searrow x} \frac{f(y)-f(x)}{y-x}$

$$\begin{aligned}
\Rightarrow f'(0+) &= \lim_{y \searrow 0} \frac{f(y)-f(0)}{y} \\
&= \lim_{y\searrow 0} \frac{1}{y}\left(\sum_{n=1}^{\infty} a_n e^{-ny} - a_n\right) \\
&= \lim_{y \searrow 0} \frac{1}{y} \sum_{n=1}^{\infty} a_n (1-e^{-ny}) \\
&\overset{*}{=} \sum_{n=1}^{\infty} a_n \lim_{y\searrow 0} \frac{1}{y}(1-e^{-ny}) \\
&\overset{*}{=} \sum_{n=1}^{\infty} n a_n
\end{aligned}$$

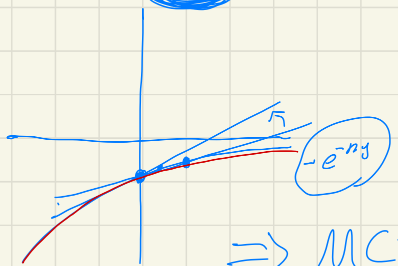
$\lim_{y\searrow 0} \frac{1}{y} \sum_{n=1}^{\infty} a_n (e^{-ny}-1)$

$= \sum_1^{\infty} a_n \lim_{y\searrow 0} \frac{1}{y}(e^{-ny}-1)$

$= \sum_1^{\infty} a_n n \quad$ exists.

* DCT on $f_n(y) = \frac{1}{y} a_n (1-e^{-ny})$

$\frac{d}{dy} -e^{-ny}\Big|_0 = \lim_{y\to 0} \frac{-e^{-ny}+1}{y} = \lim_{y\to 0} \frac{1-e^{-ny}}{y}$



$\Rightarrow$ MCT.

If $\sum_{n=1}^{\infty} n a_n < \infty$, then

$\frac{d}{dn}(1-e^{-ny}) = ye^{-ny} > 0$

$\frac{1}{y} a_n (1-e^{-ny}) \leq \frac{1}{y} a_n$

bounded for $y > \varepsilon$.

$$\begin{aligned}
\frac{d}{dy} \frac{1}{y} a_n (1-e^{-ny}) &= \left(-\frac{1}{y^2}\right) a_n (1-e^{-ny}) + \frac{1}{y} a_n n e^{-ny} \\
&= \frac{1}{y} a_n e^{-ny}\left(\frac{1}{y}+n\right) - \frac{1}{y^2} a_n \\
&= \frac{1}{y} a_n \left(e^{-ny}\left(\frac{1}{y}+n\right) - \frac{1}{y}\right) < 0.
\end{aligned}$$

$f_n \downarrow$ as $y \nearrow$.

$\frac{1}{y} > e^{-ny}\left(\frac{1}{y}+n\right)$

$1 > \underbrace{e^{-ny}}_{\leq 1} \underbrace{(1+yn)}_{\geq 1} \iff e^{ny} > 1+yn$

$e^x > 1+x$? Yes.

* $\lim_{y\searrow 0} \frac{1}{y}(1-e^{-ny}) = \lim_{y\searrow 0} n e^{-ny} = n.$

$\frac{1}{x}(1-e^{-x})$

$\frac{d}{dx} = \frac{xe^{-x}-(1-e^{-x})}{x^2}$

$= \frac{e^{-x}(1+x)-1}{x^2} = 0$

$e^x - 1 = x \qquad 1+x = e^x \qquad e^{-x} = \frac{1}{1+x}$

Note $f'(x+) = \lim_{y \searrow x} \frac{f(y) - f(x)}{y - x}$

$$\Rightarrow f'(0+) = \lim_{y \searrow 0} \frac{f(y) - f(0)}{y}$$
$$= \lim_{y \searrow 0} \frac{1}{y} \left(\sum_{n=1}^{\infty} a_n e^{-ny} - a_n \right)$$
$$= \lim_{y \searrow 0} \frac{1}{y} \sum_{n=1}^{\infty} a_n (1 - e^{-ny})$$
$$= \sum_{n=1}^{\infty} a_n \lim_{y \searrow 0} \frac{1}{y} (1 - e^{-ny})$$
$$= \sum_{n=1}^{\infty} n\, a_n$$

e^{-nh} close to 1.