19 Fa.1

1. Let f be a real valued function on a set X. Let $\mathcal{M}$ be the smallest σ-algebra on X for which f is measurable. Prove that $\{x\} \in \mathcal{M}$ for all $x \in X$ if and only if f is one-to-one.

First, assume f is one-to-one.
f measurable $\Rightarrow$ $f^{-1}(U) \in \mathcal{M}$ for U measurable in $\mathbb{R}$.

Let $x \in X$. f one-to-one $\Rightarrow$ $f^{-1}(f(x)) = \{x\}$.
$\{f(x)\}$ is measurable in $\mathbb{R}$, and f measurable $\Rightarrow$ $\{x\} \in \mathcal{M}$.

Now assume $\{x\} \in \mathcal{M}$ $\forall x \in X$. Let x, y be two points in X s.t. $f(x) = f(y)$.

$\mathcal{M}' = \{E \in \mathcal{M} : x, y \in E \text{ OR } x, y \notin E\}$. Note $x, y \in E$ means Both.

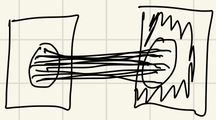
$E \in \mathcal{M}' \Rightarrow E^c \in \mathcal{M}'$ since $x, y \in E \iff x, y \notin E^c$.

$\{E_i\}_1^\infty \subset \mathcal{M}'$.

$\{f^{-1}(A) : A \in \mathcal{B}_{\mathbb{R}}\}$
σ-algebra?
$f^{-1}(A^c) = f^{-1}(A)^c$?

If $x, y \notin E_i$ $\forall i$, then $x, y \notin \bigcup E_i$.

Otherwise $\exists i$ s.t. $x, y \in E_i \Rightarrow x, y \in \bigcup E_i$.



$X, \emptyset \in \mathcal{M}'$ as well.

$\Rightarrow \mathcal{M}'$ is a σ-algebra $\Rightarrow \mathcal{M} \subset \mathcal{M}'$ by minimality.

$\Rightarrow \{x\}, \{y\} \in \mathcal{M}'$ a contradiction.

Thus f must be one-to-one.

Is f measurable for $\mathcal{M}'$? $f^{-1}((b, \infty)) \ni x, y$ if $a \in (b, \infty)$
$\not\ni x, y$ if $a \notin (b, \infty)$

$\Rightarrow f^{-1}((b, \infty)) \in \mathcal{M}'$ $\forall b \in \mathbb{R} \Rightarrow f$ measurable.

19Fa.2

2. Let m be the Lebesgue measure on $[0,1]$. Let $f_n\colon [0,1] \to \mathbb{R}$ be measurable functions. Assume that $\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx \le 1$. Prove that $f_n \to 0$ as $n \to \infty$ almost everywhere.

? Seems too short.

$$\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\,dx \overset{*}{=} \int_0^1 \sum_{n=1}^{\infty} |f_n(x)|\,dx \le 1$$

$\Rightarrow \sum_{n=1}^{\infty} |f_n(x)|$ converges almost everywhere

$\Rightarrow |f_n(x)| \longrightarrow 0 \Rightarrow f_n(x) \longrightarrow 0$ almost everywhere.

* We can exchange $\sum_{n=1}^{\infty}$ and $\int_0^1$ since $|f_n(x)| \in L^+$.

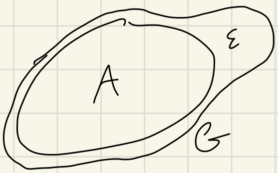
$$\begin{aligned} \sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\,dx &= \lim_{k\to\infty} \sum_{n=1}^{k} \int_0^1 |f_n(x)|\,dx \\ &= \lim_{k\to\infty} \int_0^1 \sum_{n=1}^{k} |f_n(x)|\,dx \\ &= \int_0^1 \lim_{k\to\infty} \sum_{n=1}^{k} |f_n(x)|\,dx \quad \text{by MCT.} \end{aligned}$$

19Fa.3

3. Let m be the Lebesgue measure on $[0,1]$. Prove that there does not exists a measurable set $A \subseteq [0,1]$ such that $m(A \cap [a,b]) = (b-a)/2$ for all $0 \le a < b \le 1$.

? Zejing's disjoint union of closed intervals.

Let G open $\supset A$ s.t. $m(G \setminus A) < \varepsilon$.



$G = \bigsqcup_1^\infty (a_i, b_i)$

$G \setminus A = \left(\bigsqcup_1^\infty (a_i, b_i)\right) \setminus A = \bigsqcup_1^\infty \left((a_i, b_i) \setminus A\right)$

$$\begin{aligned} m((a_i, b_i) \setminus A) &= m\left((a_i, b_i) \setminus (A \cap (a_i, b_i))\right) \\ &= m((a_i, b_i)) - m(A \cap (a_i, b_i)) \\ &= (b_i - a_i) - \frac{b_i - a_i}{2} \\ &= \frac{b_i - a_i}{2}. \end{aligned}$$

Then, $m(G \setminus A) = \sum_1^\infty m((a_i, b_i) \setminus A) = \sum_1^\infty \frac{b_i - a_i}{2} = \frac{1}{2} m(G)$

$\Rightarrow \quad \frac{1}{2} m(G) < \varepsilon$

$\Rightarrow \quad m(A) < m(G) < 2\varepsilon$, which is a contradiction since

$m(A) = m(A \cap [0,1]) = \frac{1}{2}$ and $\frac{1}{2} \ge 2\varepsilon$ for small ε.

19Fa.4

4. Let $V_f(0,x)$ be the total variation of f on $[0,x]$. Prove that if $f(x)$ is absolutely continuous on $[0,1]$, then so is $V_f(0,x)$.

$$V_f(0,x) = \sup_{0=a_1<a_2<\cdots<a_n=x} \sum |f(a_{i+1}) - f(a_i)|$$

f abs. cont. $\Rightarrow$ f differentiable a.e. on $[0,1]$ and $f' \in L^1([0,1])$ with $\int_a^b f'(x)\,dx = f(b) - f(a)$ for $0 \le a \le b \le 1$.

$V_f(0,x)$ increasing $\Rightarrow$ discontinuities are at most countable $\Rightarrow$ $V_f'(0,x)$ defined a.e. on $[0,1]$.

$\exists$ sequence $0 = a_1 < \cdots < a_n = x$ s.t. $\sum |f(a_{i+1}) - f(a_i)| > V_f(0,x) - \varepsilon$.
Then, for this sequence,
$\int_0^x |f'(t)|\,dt = \sum \int_{a_i}^{a_{i+1}} |f'(t)|\,dt \le \sum \left| \int_{a_i}^{a_{i+1}} f'(t)\,dt \right| = \sum |f(a_{i+1}) - f(a_i)| > V_f(0,x) - \varepsilon$.
Since ε arbitrary, $\int_0^x |f'(t)|\,dt \ge V_f(0,x)$.

For any $x<y \in [0,1]$, we have $|f(y) - f(x)| \le V_f(y) - V_f(x)$
Then for $x \in [0,1]$ where $f'(x)$ and $V_f'(0,x)$ exist,
we have $|f'(x)| = \left| \lim_{y\to x} \frac{f(y)-f(x)}{y-x} \right| = \lim_{y\to x} \frac{|f(y)-f(x)|}{|y-x|} \le \lim_{y\to x} \frac{V_f(y)-V_f(x)}{|y-x|} = V_f'(0,x)$.

$\Rightarrow$ $\int_0^x |f'(t)|\,dt \le \int_0^x V_f'(0,t)\,dt \overset{*}{\le} V_f(0,x)$

* since V_f increasing.

? This right.

Hence $\int_0^x |f'(t)|\,dt = V_f(0,x)$ and $|f'(t)| \in L^1([0,1])$
$\Rightarrow$ $V_f(0,x)$ absolutely continuous.