REAL ANALYSIS GRADUATE EXAM

Fall 2019

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let f be a real valued function on a set X. Let $\mathcal{M}$ be the smallest σ-algebra on X for which f is measurable. Prove that $\{x\} \in \mathcal{M}$ for all $x \in X$ if and only if f is one-to-one.

2. Let m be the Lebesgue measure on $[0,1]$. Let $f_n \colon [0,1] \to \mathbb{R}$ be measurable functions. Assume that $\sum_{n=1}^{\infty} \int_0^1 |f_n(x)|\, dx \leq 1$. Prove that $f_n \to 0$ as $n \to \infty$ almost everywhere.

3. Let m be the Lebesgue measure on $[0,1]$. Prove that there does not exists a measurable set $A \subseteq [0,1]$ such that $m(A \cap [a,b]) = (b-a)/2$ for all $0 \leq a < b \leq 1$.

4. Let $V_f(0,x)$ be the total variation of f on $[0,x]$. Prove that if $f(x)$ is absolutely continuous on $[0,1]$, then so is $V_f(0,x)$.