18 Fa. 1

Problem 1. Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function. Let

$$g(x) = \int_0^1 f(xt)dt, \qquad x \in [0,1].$$

Show that g is an absolutely continuous function.

f absolutely continuous $\Rightarrow$ for $0 \le a \le b \le 1$, $\exists$ $h \in L^1$ s.t. $f(b) - f(a) = \int_a^b h(t)dt$.

NTS $\exists$ $\bar{g} \in L^1$ s.t. $g(x) - g(0) = \int_0^x \bar{g}\, d\cancel{t}$.

$$\begin{aligned} g(x) - g(0) &= \int_0^1 f(xt)dt - \int_0^1 f(0)dt \\ &= \int_0^1 [f(xt) - f(0)]dt \\ &= \int_0^1 \int_0^{xt} h(s)ds\,dt \\ &= \int_0^1 \int_0^1 \chi_{[0,xt]}(s) h(s) ds\,dt \\ &= \int_0^1 \int_0^1 \chi_{[s/x,1]}(t) h(s) dt\,ds \\ &= \int_0^1 h(s)(1 - s/x) ds \end{aligned}$$

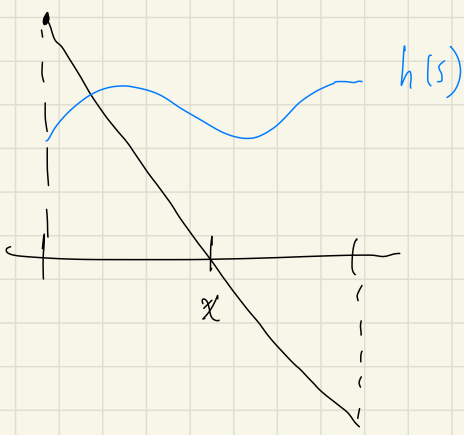
$s \in [0, xt]$

$0 \le s \le xt$

$1 \ge t \ge \frac{s}{x}$

$x = 0?$

$g(x) = g(0)$ ✓



$u = 1 - \frac{s}{x}$ $\quad s = (1-u)x$

$du = -\frac{1}{x}$

$1 - \frac{1}{x} = \frac{x-1}{x}$

18Fa.2

Problem 2. Let $f \in L^1(\mathbb{R})$, and let

$$S_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} f\left(x + \frac{j}{n}\right), \qquad x \in \mathbb{R},$$

$$S(x) = \int_x^{x+1} f(y)dy, \qquad x \in \mathbb{R}.$$

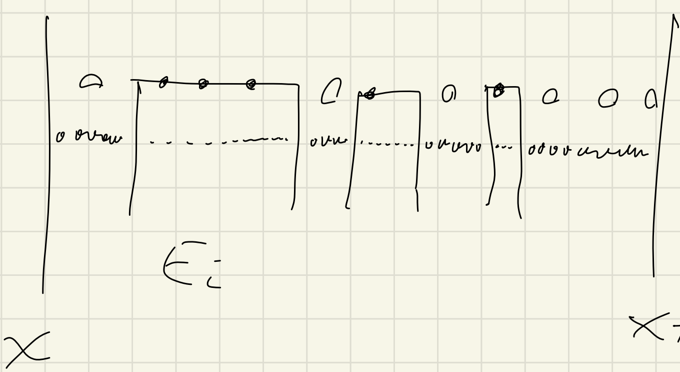
Show that $f_n \to f$ in $L^1(\mathbb{R})$.

> ***We assume that the question meant to say "Show that $S_n \to S$ in $L^1(\mathbb{R})$" since no f_n is ever defined.

$$\begin{aligned}\int |S_n(x) - S(x)|\,dx &= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} f(x+\tfrac{j}{n}) - \int_x^{x+1} f(y)\,dy \right| dx \\ &= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} f(x+\tfrac{j}{n}) - \sum_{j=0}^{n-1}\int_{x+j/n}^{x+(j+1)/n} f(y)\,dy \right| dx \\ &= \int_{\mathbb{R}} \left| \sum_{j=0}^{n-1} \frac{1}{n} f(x+\tfrac{j}{n}) - \int_{x+j/n}^{x+(j+1)/n} f(y)\,dy \right| dx \\ &= \int_{\mathbb{R}} \left| \sum_{j=0}^{n-1} \int_{x+j/n}^{x+(j+1)/n} f(x+j/n) - f(y)\,dy \right| dx\end{aligned}$$

First, we prove the result for a simple function, $\phi = \sum_1^k \alpha_i \chi_{E_i}$

$$\begin{aligned}\int_{\mathbb{R}} |S_n(x) - S(x)|\,dx &= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} \sum_{i=1}^{k} \alpha_i \chi_{E_i}(x+\tfrac{j}{n}) - \int_x^{x+1} \sum_{i=1}^{k} \alpha_i \chi_{E_i}(y)\,dy \right| dx \\ &= \int_{\mathbb{R}} \left| \frac{1}{n}\sum_{j=0}^{n-1} \sum_{i=1}^{k} \alpha_i \chi_{E_i}(x+\tfrac{j}{n}) - \sum_{i=1}^{k} \alpha_i\, m(E_i \cap [x,x+1]) \right| dx \\ &= \int_{\mathbb{R}} \left| \sum_{i=1}^{k} \alpha_i \left[\frac{1}{n}\sum_{j=0}^{n-1} \chi_{E_i}(x+\tfrac{j}{n}) - m(E_i \cap [x,x+1]) \right] \right| dx\end{aligned}$$



18 Fa. 3

Problem 3. Assume that f_n is a sequence of integrable functions on $\mathbb{R}$ such that

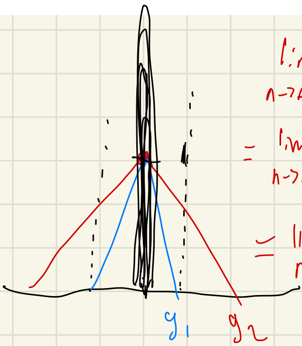
$$\lim_{n\to\infty}\int f_n(x)g(x)dx = g(0)$$

for all g continuous with compact support.

Prove that f_n is not a Cauchy sequence in $L^1(\mathbb{R})$.

Kaylen

wrong?

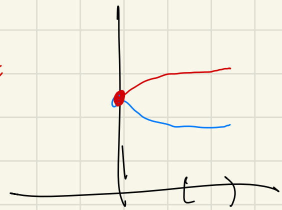


$$\lim_{n\to\infty}\int f_n(x)g(x)\,dx$$

$$= \lim_{n\to\infty}\int_{\{f_n(x)\to 0\}} f_n(x)g(x)\,dx + \int_{\{f_n \not\to 0\}} f_n g\,dx$$

$$= \lim_{n\to\infty}\int_{f_n(x)\to 0} f_n\tilde{g}\,dx + \int_{f_n\not\to 0} f_n\tilde{g}\,dx$$

$$\int_{[\varepsilon,\varepsilon]} f_n g \not\to 0.$$



$$\lim_{n\to\infty}\int f_n(x)g_1(x)\,dx \approx \lim_{n\to\infty}\int_0^1 f_n(x)\,dx$$

$$\lim_{n\to\infty}\int f_n(x)g_2(x)\,dx \approx \lim_{n\to\infty}\int_{-1}^0 f_n(x)\,dx$$

$= 1$ doesn't go to 0

$$\lim_{n\to\infty}\int f_n(x)g_1(x)\,dx = \lim_{n\to\infty}\int f_n(x)g_2(x)\,dx = 1$$

$$\lim_{n\to\infty}\int_{-1}^1 f_n(x)g_1(x)\,dx = \lim_{n\to\infty}\int_{-2}^2 f_n(x)g_2(x)\,dx$$

$f_n = \chi_{[-\frac{1}{n},\frac{1}{n}]}\, n/2$ Cauchy? $f_n \to 0$ a.e. but $\int f_n - 0 \not\to 0$.

$2/n$ $n/2$

$$\lim_{n\to\infty}\int f_n g = \lim_{n\to\infty}\int \frac{n}{2}\chi_{[-\frac{1}{n},\frac{1}{n}]}\, g$$

$=$