17Fa.1

Problem 1. Let $(X, \mathcal{A}, \mu)$ be a measure space and f, g, f_n, g_n measurable so that $f_n \to f$ and $g_n \to g$ in measure. Is it true that $f_n^3 + g_n \to f^3 + g$ in measure if

(a) $\mu(X) = 1$

(b) $\mu(X) = \infty$

In both cases prove the statement or provide a counter example.

a) Since $\mu(X) < \infty$, $\exists\ M > 1$ s.t. $\mu(\{x : |f| \geq M\}) < \varepsilon$. ? True ?

$\exists\ N$ s.t. $\forall\ n \geq N$, $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon/12M^2\}) < \varepsilon$
and $\mu(\{x : |g_n(x) - g(x)| \geq \varepsilon/4\}) < \varepsilon$

If $|f(x)| < M$, $|f_n(x) - f(x)| < \varepsilon/12M^2$, and $|g_n(x) - g(x)| < \varepsilon/4$

$$
\begin{aligned}
|f_n^3(x) - f^3(x) + g_n(x) - g(x)| &\leq |f_n^3(x) - f^3(x)| + |g_n(x) - g(x)| \\
&= |(f_n - f)^3 + 3f_nf(f_n - f)| + |g_n - g| \\
&= |(f_n - f)^3 + 3(f + (f_n - f))f(f_n - f)| + |g_n - g| \\
&= |(f_n - f)^3 + 3f^2(f_n - f) + 3f(f_n - f)^2| + |g_n - g| \\
&\leq |f_n - f|^3 + 3M^2|f_n - f| + 3M|f_n - f|^2 + |g_n - g| \\
&< \varepsilon^3/12^3M^6 + 3M^2\,\varepsilon/12M^2 + 3M\,\varepsilon^2/12^2M^4 + \varepsilon/4 \\
&\leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \\
&= \varepsilon.
\end{aligned}
$$

Thus, $\{x : |f_n^3 + g_n - f^3 - g| \geq \varepsilon\} \subset \{x : |f_n - f| \geq \varepsilon/12M^2\} \cup \{x : |g_n - g| \geq \varepsilon/4\} \cup \{x : |f| \geq M\}$

$$
\begin{aligned}
\Rightarrow \quad \mu(\{x : |f_n^3 + g_n - f^3 - g| \geq \varepsilon\}) &\leq \mu(\{x : |f_n - f| \geq \varepsilon/12M^2\}) + \mu(\{x : |g_n - g| \geq \varepsilon/4\}) \\
&\quad + \mu(\{x : |f| \geq M\}) \\
&< 3\varepsilon
\end{aligned}
$$

Since ε arbitrary, $f_n^3 + g_n \longrightarrow f^3 + g$ in measure.

$\square$

b) Consider $f_n(x) = x + \frac{1}{n}$, $f(x) = x$ on $(\mathbb{R}, \mathcal{M}, m)$.

Let $\varepsilon > 0$. $m(\{x : |f_n(x) - f(x)| \geq \varepsilon\})$

$= m(\{x : \frac{1}{n} \geq \varepsilon\}) \longrightarrow 0$ as $n \to \infty$.

Thus, $f_n \longrightarrow f$ in measure.

But, $f_n^3(x) = x^3 + 3x^2\frac{1}{n} + 3x\frac{1}{n^2} + \frac{1}{n^3}$.

Let $\varepsilon > 0$. $m(\{x : |f_n^3(x) - f^3(x)| \geq \varepsilon\})$

$= m(\{x : 3x^2\frac{1}{n} + 3x\frac{1}{n^2} + \frac{1}{n^3} \geq \varepsilon\}) = \infty$

for each n, since $\forall\, x \geq n\sqrt{\varepsilon}$,

we have $3x^2\frac{1}{n} + 3x\frac{1}{n^2} + \frac{1}{n^3} \geq 3n\varepsilon + 3\sqrt{\varepsilon}\frac{1}{n} + \frac{1}{n^3} \geq \varepsilon$.

Take $g_n = g$ $\forall\, n$, and the counterexample is shown.

17Fa.2

Problem 2. Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} f(x+n)$$

converges absolutely for Lebesgue almost every $x \in \mathbb{R}$.

For $M \in \mathbb{Z}$,
$$\int_M^{M+1} \sum_{n=1}^{\infty} |f(x+n)|\,dx = \sum_{n=1}^{\infty} \int_M^{M+1} |f(x+n)|\,dx \quad *$$
$$= \sum_{n=1}^{\infty} \int_{M+n}^{M+n+1} |f(x)|\,dx$$
$$= \int_{M+1}^{\infty} |f(x)|\,dx$$
$$< \infty$$

Thus, for almost every $x \in [M, M+1]$, $\sum_{n=1}^{\infty} |f(x+n)| < \infty$. And since M was arbitrary, this is true for almost every $x \in \mathbb{R}$.

$\square$

$*$ We can pull the sum out of the integral because $|f(x+n)| \in L^+$.

17Fa.3

Problem 3. Assume that $E \subset \mathbb{R}$ is such that $m(E \cap (E+t)) = 0$ for all $t \neq 0$, where m is the Lebesgue measure on $\mathbb{R}$. Prove that $m(E) = 0$.

$$
\begin{aligned}
m(E \cap (E+t)) &= \int \chi_E \chi_{E+t} \, dm \\
&= \int \chi_E(x) \chi_E(x-t) \, dm(x) \\
&= 0 \qquad \forall\, t \neq 0.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \int \int \chi_E \chi_{E+t} \, dm(x) \, dm(t) = 0 \\
&= \int \int \chi_E \chi_{E+t} \, dm(t) \, dm(x) \\
&= \int \chi_E \int \chi_{E+t} \, dm(t) \, dm(x) \\
&= \int \chi_E(x) \int \chi_{x-E}(t) \, dm(t) \, dm(x) \\
&= \int \chi_E(x) \, m(x-E) \, dm(x) \\
&= m(E) \int \chi_E(x) \, dm(x) \\
&= m(E)^2 \\
&= 0 \\
\Rightarrow \quad & m(E) = 0.
\end{aligned}
$$

17 Fa. 4

Problem 4. Let $(X, \mathcal{A}, \mu)$ be a measure space and f_n a sequence of non-negative measurable functions. Prove that if $\sup_n f_n$ is integrable, then

$$\limsup_n \int_X f_n d\mu \leq \int_X \limsup_n f_n d\mu.$$

Also show that

(a) the inequality may be strict and

(b) that the inequality may fail unless $\sup_n f_n \in L^1$.

For fixed $K \in \mathbb{N}$,

$$f_n \leq \sup_{n \geq K} f_n \quad \forall \; n \geq K$$

$$\Rightarrow \int_X f_n \, d\mu \leq \int_X \sup_{n \geq K} f_n \, d\mu \quad \forall \; n \geq K$$

$$\Rightarrow \sup_{n \geq K} \int_X f_n \, d\mu \leq \int_X \sup_{n \geq K} f_n \, d\mu$$

Thus, $\lim_{K \to \infty} \sup_{n \geq K} \int_X f_n \, d\mu \leq \lim_{K \to \infty} \int_X \sup_{n \geq K} f_n \, d\mu = \int_X \lim_{K \to \infty} \sup_{n \geq K} f_n \, d\mu$

by DCT, since $\sup_{n \geq K+1} f_n \leq \sup_{n \geq K} f_n$ and $\sup_{n \geq 1} f_n$ integrable. $\square$

a) Consider $f_n = \begin{cases} \chi_{[0, 1/2]} & \text{if } n \text{ odd} \\ \chi_{[1/2, 1]} & \text{if } n \text{ even.} \end{cases}$

? These examples good ?

Then $\limsup_n \int_0^1 f_n \, d\mu = \limsup_n \frac{1}{2} = \frac{1}{2}$

And $\int_0^1 \limsup_n f_n \, d\mu = \int_0^1 1 \, d\mu = 1$.

b) Consider $f_n = \chi_{[n, n+1]}$, $\sup_n f_n = \chi_{[0, \infty]} \notin L^1$.

$\limsup_n \int_0^\infty f_n \, d\mu = \limsup_n 1 = 1$.

$\int_0^\infty \limsup_n f_n \, d\mu = \int_0^\infty 0 \, d\mu = 0$.

Simpler than Kaplas.