16 Fa. 1

Problem 1. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of *nonnegative* measurable functions. Prove that $f_n \to 0$ in measure if and only if

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

First, assume $f_n \to 0$ in measure.
Then for $\varepsilon > 0$, $\mu(\{x : |f_n(x)| \geq \varepsilon\}) \longrightarrow 0$ as $n \to \infty$.
Call these sets E_n.
Then
$$\int \frac{f_n}{f_n+1} d\mu = \int_{E_n} \frac{f_n}{f_n+1} d\mu + \int_{E_n^c} \frac{f_n}{f_n+1} d\mu$$
$$\leq \int_{E_n} 1 \, d\mu + \int_{E_n^c} \varepsilon \, d\mu$$
$$\leq \mu(E_n) + \mu(X)\varepsilon$$

So $\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu \leq \mu(X)\varepsilon$. And since ε was arbitrary,

$$\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0.$$

$\square$

Second, assume $\lim_{n\to\infty} \int \frac{f_n}{f_n+1} d\mu = 0$.
Then $\exists\, N \in \mathbb{N}$ s.t. $\forall\, n \geq N$, $\int \frac{f_n}{f_n+1} d\mu < \varepsilon$, for a given $\varepsilon > 0$.
Assume $f_n \not\to 0$ in measure. Then for some $\varepsilon > 0$,
$\mu(\{x : |f_n(x)| \geq \varepsilon\}) \not\to 0$ as $n \to \infty$. Call these sets E_n.

We know that
$$\frac{d}{dx} \frac{x}{x+1} = \frac{(x+1) - x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0 \quad \text{for } x \neq -1$$
$$\Rightarrow \frac{f_n}{f_n+1} \geq \frac{\varepsilon}{\varepsilon+1} \quad \text{whenever } f_n \geq \varepsilon.$$

Thus, $\int \frac{f_n}{f_n+1} d\mu \geq \int_{E_n} \frac{\varepsilon}{\varepsilon+1} d\mu = \mu(E_n) \frac{\varepsilon}{\varepsilon+1} \not\to 0$ as $n \to \infty$.

This is a contradiction, so $f_n \to 0$ in measure.

$\square$

16 Fa. 2

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of sets. Assume that $\mu(A_n) \geq \delta$ for all $n \in \mathbb{N}$, where $\delta > 0$. Prove that there exists a set $S \in \mathcal{F}$ of positive measure such that for every $x \in S$, is in A_j for infinitely many j.

Define $S := \{ x \in X : x \in A_j \text{ for infinitely many } j \}$.

Define $S_K := \{ x \in X : x \in A_j \text{ for exactly } K \text{ different } j \}$, $K = 0, 1, 2, \dots$

Assume $\mu(S) = 0$.

We know $\sum_{K=0}^{\infty} \mu(S_K) = \mu(X)$

$f(x) = \sum_j \chi_{A_j}$ NTS $f(x) = \infty$ on non null set.

$f = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n}$

$$\int \sum_{n=1}^{\infty} \frac{1}{n} \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n} \int \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n} \mu(A_n) \geq \sum_{n=1}^{\infty} \frac{1}{n} \delta = \infty.$$

$$\mu(X) \geq \int \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} \frac{1}{2^n} \mu(A_n) \geq \delta$$

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16 Fa. 3

Problem 3. Let $f_n : [0,1] \to [0,\infty)$ be Lebesgue measurable and such that $f_n(x) \to 0$ for almost every x. Assume that

$$\sup_n \int_0^1 \varphi(f_n(x))dx \le 1$$

for some continuous $\varphi : [0,\infty) \to [0,\infty)$ which satisfies $\varphi(t)/t \to \infty$ as $t \to \infty$. Prove that $\int_0^1 f_n(x)dx \to 0$ as $n \to \infty$. (Provide a detailed proof).

$\varphi(t)/t \longrightarrow \infty$ as $t \to \infty \implies$ for $M \in \mathbb{N}$, $\exists\ T \in \mathbb{N}$ s.t.
$\forall\ t \geq T$, $\varphi(t)/t \geq M$.

Define $E_n^T := \{x : f_n(x) \geq T\}$

16 Fa. 4

Problem 4. Let $h : [0,\infty) \to \mathbb{R}$ be continuous with compact support. Prove that

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{h(\alpha x)-h(\beta x)}{x}\,dx = h(0)\log\frac{\alpha}{\beta}$$

for every $\alpha,\beta>0$.

? DCT

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First, take $\beta \geq \alpha$.

$$\lim_{\varepsilon\to 0^+}\int_\varepsilon^\infty \frac{h(\alpha x)-h(\beta x)}{x}\,dx = \lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\int_\varepsilon^M \frac{h(\alpha x)}{x}\,dx - \int_\varepsilon^M \frac{h(\beta x)}{x}\,dx$$

$$= \lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\int_{\alpha\varepsilon}^{\alpha M} \frac{h(u)}{u}\,du - \int_{\beta\varepsilon}^{\beta M}\frac{h(u)}{u}\,du$$

$$= \lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\int_{\alpha\varepsilon}^{\beta\varepsilon} \frac{h(u)}{u}\,du - \int_{\alpha M}^{\beta M}\frac{h(u)}{u}\,du$$

~~$u = x/\varepsilon$~~ ~~$x = u\varepsilon$~~ $du =$

$$= \lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\int_\alpha^\beta \frac{h(\overset{x\varepsilon}{x/\varepsilon})}{x}\,dx - \int_\alpha^\beta \frac{h(\overset{xM}{x/M})}{x}\,dx$$

$x = u/\varepsilon$ $\quad u = x\varepsilon$, $du = \varepsilon\, dx$

$$= \lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\int_\alpha^\beta \frac{h(\overset{x\varepsilon}{x/\varepsilon}) - h(\overset{xM}{x/M})}{x}\,dx$$

$$= \int_\alpha^\beta \lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\frac{h(x/\varepsilon)-h(x/M)}{x}\,dx \qquad *$$

$$= \int_\alpha^\beta \frac{0^* - h(0)}{x}\,dx$$

$$= -h(0)\left(\ln(\beta) - \ln(\alpha)\right) = h(0)\ln\left(\frac{\alpha}{\beta}\right).$$

If $\alpha \geq \beta$, we have $\lim_{\varepsilon\to 0^+}\lim_{M\to\infty}\int_{\beta M}^{\alpha M}\frac{h(u)}{u}\,du - \int_{\beta\varepsilon}^{\alpha\varepsilon}\frac{h(u)}{u}\,du$

$$= \int_\beta^\alpha \frac{h(0)}{x}\,dx = h(0)\left(\ln(\alpha)-\ln(\beta)\right) = h(0)\ln\left(\frac{\alpha}{\beta}\right).$$

$\square$

* DCT

* $h(x/\varepsilon) \to 0$ as $\varepsilon \to 0$ b/c h has compact $\Rightarrow$ bounded support.