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Real Analysis Exam Fall 2015

Problem 1. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form p/q such that $q \geq 2$ and $|x - p/q| < 1/(q\log q)^2$. (Hint: Consider intervals of length $2/(q\log q)^2$ centered at rational points p/q).

Solution. Let $Q(x) = \{p/q \in \mathbb{Q} \,|\, (p,q) = 1, q \geq 2, |x - p/q| < 1/(q\log q)^2\}$.

Then $Q(x)$ counts the number of such rationals stated in the problem. Note that if $\int_{\mathbb{R}} Q(x)dx < \infty$, then $Q(x)$ must be finite a.e.

Now, since

$$\begin{aligned}
\int_{\mathbb{R}} Q(x)dx &= \sum_{q\geq 2\in\mathbb{N}} \int_{B(1/(q\log q)^2, p/q)} Q(x)dx \\
&= \sum_{q=2}^{\infty} (q-1)m(B(1/(q\log q)^2, p/q)) \qquad (1) \\
&= \sum_{q=2}^{\infty} \frac{2(q-1)}{q^2\log^2 q} \\
&= 2\sum_{q=2}^{\infty} \frac{1}{q\log^2 q} - 2\sum_{q=2}^{\infty} \frac{1}{q^2\log^2 q}
\end{aligned}$$

$\boxed{(1)}$ Because the Lebesgue measure is translation invariant, we may take $p < q$. Since the number of rationals in an interval is also invariant under translation, if $p > q$, then p is an integer shift of some $p' < q$.

The $(q-1)$ comes from the number of integers $1 < p < q$.

Now, we check the convergence of both sums.

$\boxed{\sum_{q=2}^{\infty} \frac{1}{q^2\log^2 q}}$ converges by limit comparison test. Specifically,

$$\lim_{q\to\infty} \frac{1}{q^2\log^2 q} \Big/ \frac{1}{q^2} = \lim_{q\to\infty} \frac{1}{\log^2 q} = 0 \implies \text{ since } \sum_{q=2}^{\infty} \frac{1}{q^2} < \infty \text{ then } \sum_{q=2}^{\infty} \frac{1}{q^2\log^2 q} < \infty.$$

$\boxed{\sum_{q=2}^{\infty} \frac{1}{q\log^2 q}}$ converges by integral test.

$$\int_2^\infty \frac{1}{q\log^2 q}dq = \int_{\log 2}^\infty \frac{1}{u^2}du < \infty \qquad \text{since } \log 2 > 0$$

$$\begin{array}{cc} u = \log q & q : [2, \infty] \\ du = \frac{1}{q}dq & u : [\log 2, \infty] \end{array}$$

Thus, $\int Q(x)dx < \infty$ and so $Q(x)$ must be finite a.e. ✌

Problem 2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

Solution. First, since f is increasing, it has at most countably many discontinuities and so is measurable. Specifically, $f^{-1}((-\infty,a)) = \{x \,|\, f(x) < a\} = [0, \inf f^{-1}(a))\backslash N$ for N some null set.

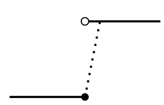
Thus, because f is measurable, there exists a sequence of simple functions $\phi_1 \leq \phi_2 \leq \cdots \leq f$ with $\phi_n \to f$ pointwise.

Thus, it suffices to show that there exists a sequence of continuous functions converging to ϕ a simple function. Furthermore, since the ϕ_n are an increasing sequence converging to an increasing function, we may take our ϕ to also be increasing.

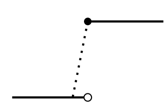
Then, let $\phi = \sum_{i=1}^m a_i \chi_{E_i}$ be the standard representation of ϕ. Since ϕ is increasing, and by definition can have only a finite set as its range, ϕ can only have a finite number of discontinuities. Furthermore, $a_i \leq a_{i+1}$ for all $1 \leq i \leq m-1$.

Now, to construct a continuous approximation to ϕ, we simply use a trapezoid approximation. Let $x_1, x_2, ..., x_{m-1}$ be the points where the jump discontinuities of ϕ occur. Now,

If $\phi(x_i) = a_i$
Let $y_i = n(a_{i+1} - a_i)(x - (x_i + \frac{1}{n}))$
$\chi_i(x) = \chi_{[x_i, x_i + \frac{1}{n}]}(x)$



If $\phi(x_i) = a_{i+1}$
$y_i = n(a_{i+1} - a_i)(x - (x_i - \frac{1}{n}))$
$\chi_i(x) = \chi_{[x_i - \frac{1}{n}, i)}(x)$



Then, the y_i are the line segments connecting the jumps of ϕ and always connected to $\phi(x_i)$.

Let

$$g_n(x) = \phi(x) + \sum_{i=1}^m y_i \chi_i(x).$$

Then, we note that $g_n(x) = \phi(x)$ for all x except within $\frac{1}{n}$ of x_i. (Note that $g_n(x_i) = \phi(x_i)$ for all x_i.

Based on our construction, it is immediate that the g_n are continuous.

Thus, if $\phi(x_i) = a_i$ and we can show that $g_n(x_i + \delta) \to \phi(x_i + \delta)$ for each x_i, and similarly for $\phi(x_i) = a_{i+1}$ and $g_n(x_i - \delta) \to \phi(x_i - \delta)$ we will be done.

Let $\varepsilon > 0$. Then, for all $\delta > 0$, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \delta$. Then, for all $n \geq N$,

$$|g_n(x_i + \delta) - \phi(x_i + \delta)| = |a_{i+1} - a_{i+1}| = 0 < \varepsilon.$$

Similarly for $x_i - \delta$.

Finally, since there are only finitely many discontinuites of ϕ, for whatever the minimum distance between any two x_i is, there exists an $N \in \mathbb{N}$ such that $\frac{1}{n}$ is less than that distance for all $n > N$. Thus, aside from possibly discarding the first finite N, $g_n \to \phi$ pointwise. ✌

Problem 3. Let (X, μ) be a finite measure space. Assume that a sequence of integrable functions f_n satisfies $f_n \to f$ in measure, where f is measurable. Assume that f_n satisfies the following property: For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\mu(E) \leq \delta \implies \int_E |f_n| d\mu \leq \varepsilon.$$

Prove that f is integrable and that

$$\lim_n \int_X |f_n - f| d\mu = 0.$$

Solution. Since $f_n \to f$ in measure, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e.

Since $f_n \in L^1(\mu)$ or all n, $E = \{x \,|\, f_n(x) = \infty\}$ is μ-null. Thus, each f_{n_k}, for any $\varepsilon > 0$ and associated δ from the problem, there exists some finite $M > 0$ such that $\mu(\{x \,|\, f_n(x) > M\} < \delta$.

Thus,

$$\begin{aligned}
\int |f| d\mu &= \int_E |f| d\mu + \int_{E^c} |f| d\mu \\
&= \int_E \liminf_{n_k} |f_{n_k} d\mu + \int_{E^c} \liminf_{n_k} |f_{n_k} d\mu \\
&\leq \liminf_{n_k} \int_E |f_{n_k} d\mu + \liminf_{n_k} \int_{E^c} |f_{n_k} d\mu \qquad \text{Fatou's Lemma} \\
&\leq \liminf_{n_k} \varepsilon + \liminf_{n_k} M\mu(X) < \infty \qquad (1)
\end{aligned}$$

(1) Since δ is from the problem, and $\mu(E) < \delta$, $\int_E |f_{n_k} d\mu \leq \varepsilon$ and $\mu(X) < \infty$.

Therefore, $f \in L^1$.

Claim 1. The above property for f_n holds for f.

Proof. Since $f \in L^1$, for the subsequence $\{f_{n_k}\}$ converging to f a.e., by Fatou's we have that, for all $\varepsilon > 0$ and δ stated in the problem, if $\mu(E) < \delta$ then

$$\int_E |f| d\mu = \int_E \liminf_{n_k} |f_{n_k}| d\mu \leq \liminf_{n_k} \int_E |f_{n_k}| d\mu \leq \liminf_{n_k} \varepsilon = \varepsilon.$$

✌

Now, let $\varepsilon > 0$ be given and δ be as from the problem. Let $F = \{x \,|\, |f_n(x) - f(x)| \geq \varepsilon\}$. Then since $f_n \to f$ in measure, there exists some N such that $\mu(F) < \delta$ for all $n \geq N$.

Then

$$\int |f_n - f| d\mu = \int_F |f_n - f| d\mu + \int_{F^c} |f_n - f| d\mu \leq 2\varepsilon + \varepsilon\mu(X).$$

Since ε is arbitrary, we have that $\int |f_n - f| d\mu \to 0$ as $n \to \infty$. ✌

Problem 4 (Folland, 2.3.25, p.59)**.** Consider the following statements about a functioin $f : [0,1] \to \mathbb{R}$.

(i) f is continuous almost everywhere

(ii) f is equal to a continuous function g almost everywhere.

Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counter example.

Solution. $\boxed{(i) \not\Longrightarrow (ii)}$ Let

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

then $f(x)$ is continuous a.e. since it is only discontinuous at $\frac{1}{2}$.

Now, assume there is some continuous function $g(x) = f(x)$ a.e.

Let $\frac{1}{2} > \varepsilon > 0$ be given. Then, by continuity of g, there exists a δ such that for all $y \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$, $|g(\frac{1}{2}) - g(y)| < \varepsilon$.

However, because $f = g$ a.e., there exists a $x_0 \in (\frac{1}{2} - \delta, \frac{1}{2})$ such that $f(x_0) = g(x_0) = 0$ and there exists $y_0 \in (\frac{1}{2}, \frac{1}{2} + \delta)$ such that $f(y_0) = g(y_0) = 1$.

however, then $|x_0 - y_0| < \delta$ and $|g(y_0) - g(x_0)| = 1 > \varepsilon$ which is a contradiction of the continuity of g.

Therefore, f is not equal to a continuous function a.e.

$\boxed{(ii) \not\Longrightarrow (i)}$ Let $f(x) = \chi_{\mathbb{Q}}$. Let $g(x) = 0$. Then $f(x) = g(x)$ a.e. (since $f(x) \neq 0$ only when $x \in \mathbb{Q}$ which is a Lebesgue-null set).

However, $f(x)$ is discontinuous at every point and so $f(x)$ is not continuous a.e.