15 Fa.1

1. Prove that for almost all $x \in [0,1]$, there are at most finitely many rational numbers with reduced form $p/q$ such that $q \geq 2$ and $|x - p/q| < 1/(q \log q)^2$. (Hint: Consider intervals of length $2/(q \log q)^2$ centered at rational points $p/q$.)

Consider the interval $\left( p/q - 1/(q\ln q)^2, \ p/q + 1/(q \ln q)^2 \right)$ for $p/q$ reduced and $q \geq 2$. Call this set $E_{p/q}$.

Consider $\sum_{p/q} \chi_{E_{p/q}}$.

$$\begin{aligned} \int \textstyle\sum_{p/q} \chi_{E_{p/q}} = \textstyle\sum_{p/q} \int \chi_{E_{p/q}} &= \textstyle\sum_{p/q} 2/(q \ln q)^2 \\ &\leq \textstyle\sum_{q=2}^{\infty} 4q/(q \ln q)^2 \quad * \\ &\leq 4 \textstyle\sum_{q=2}^{\infty} 1/q(\ln q)^2 \\ &< \infty. \end{aligned}$$

Now, consider $\{x \in [0,1] : \text{infinitely many } p/q \text{ s.t. } |x - p/q| < 1/(q \ln q)^2\}$
For $x$ in this set $\sum_{p/q} \chi_{E_{p/q}}(x) = \infty$.
Thus the measure of this set must be null since $\sum_{p/q} \chi_{E_{p/q}}$ is integrable.

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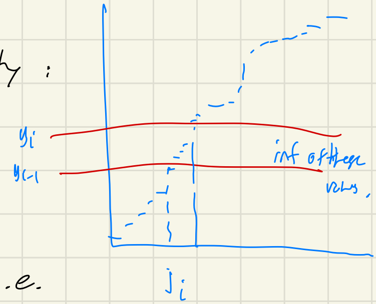
# 15 Fa. 2

2. Suppose that the real-valued function $f(x)$ is nondecreasing on the interval $[0,1]$. Prove that there exists a sequence of continuous functions $f_n(x)$ such that $f_n \to f$ pointwise on this interval.

Since $f$ nondecreasing, there are at most countably many discontinuities, $x_1, x_2, \dots$
There are 3 cases of discontinuity:
(i) $f(x-) < f(x) = f(x+)$
(ii) $f(x-) = f(x) < f(x+)$
(ii) $f(x-) < f(x) < f(x+)$

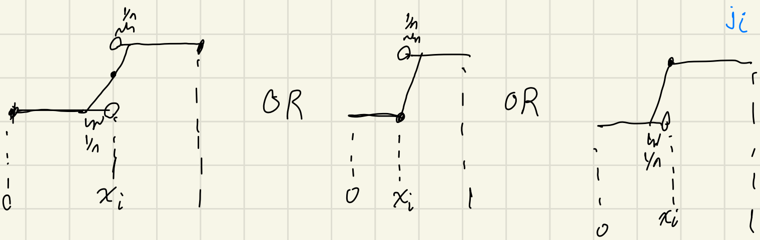



$f$ nondecreasing $\Rightarrow$ $f$ differentiable a.e.

$f_n(x) = y_{i-1}$ on $j_i$ ∀ members $j_i$

Define $g(x) = f(0) + \int_0^x f'(t)\,dt$.
This is basically $f$ without the discontinuities.

Define $h_{i,n} =$




Corresponding to the discontinuity. Define $H_n = \sum_i h_{i,n}$

Define $f_n = g + H_n$

cont. because $g$, $h_{i,n}$ continuous. & $\Sigma$ cont. is cont.

$f_n(x) \to f(x)$ ?

Simple functions converging from below, similar

15Fa. 3

3. Let $(X, \mu)$ be a finite measure space. Assume that a sequence of integrable functions $f_n$ satisfies $f_n \to f$ in measure, where $f$ is measurable. Assume that $f_n$ satisfies the following property: For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\mu(E) \le \delta \implies \int_E |f_n| \, d\mu \le \epsilon. \qquad \forall n\,?$$

Prove that $f$ is integrable and that

$$\lim_n \int_X |f_n - f| \, d\mu = 0.$$

Let $\varepsilon > 0$. Then $\exists\ \delta > 0$ s.t. $\mu(E) \le \delta \Rightarrow \int_E |f_n| \, d\mu \le \varepsilon \ \ \forall n$.
$f_n \to f$ in measure $\Rightarrow \exists$ subsequence $f_{n_k} \to f$ a.e.
And since $\mu(X) < \infty$, Egoroff $\Rightarrow \exists$ set $E$ s.t.
$f_{n_k} \to f$ uniformly on $E$ and $\mu(E^c) < \delta$. Check?
$\Rightarrow \int_{E^c} |f_n| \, d\mu \le \varepsilon \ \ \forall n$.
Then $\int_{E^c} |f| \, d\mu = \int_{E^c} \liminf |f_{n_k}| \, d\mu \le \liminf \int_{E^c} |f_{n_k}| \, d\mu \le \varepsilon$.
$f_{n_k} \to f$ uniformly on $E \Rightarrow \exists\ N_K \in \{n_k\}$ s.t. $|f_{N_K}(x) - f(x)| < \varepsilon$
$\forall x \in E$. Then $\int_E |f| \, d\mu < \int_E |f_{N_K}| + \varepsilon \, d\mu < \infty$
Thus, $\int_X |f| \, d\mu = \int_E |f| \, d\mu + \int_{E^c} |f| \, d\mu < \infty$.

$\square$

Now, let $C := \int_X |f| \, d\mu$.
$f_n \to f$ in measure $\Rightarrow$ for $\varepsilon > 0$ $\mu(\{x : |f_n(x) - f(x)| \ge \varepsilon\}) \to 0$ as $n \to \infty$.
Call this set $E_n$. Then $\exists\ N \in \mathbb{N}$ s.t. $\forall n \ge N$, we have
$\mu(E_n) \le \delta \Rightarrow \forall n, \ \int_{E_n} |f_n| \, d\mu \le \varepsilon$.
Thus, $\forall n \ge N$,

$$\begin{aligned}
\int_X |f_n - f| \, d\mu &= \int_{E_n} |f_n - f| \, d\mu + \int_{E_n^c} |f_n - f| \, d\mu \\
&\le \int_{E_n} |f_n| \, d\mu + \int_{E_n} |f| \, d\mu + \varepsilon \mu(X) \\
&\le \varepsilon + \int_{E_n} \liminf |f_{n_k}| \, d\mu + \varepsilon \mu(X) \\
&\le \varepsilon + \varepsilon + \varepsilon \mu(X)
\end{aligned}$$

Since $\varepsilon$ arbitrary, $\lim_n \int_X |f_n - f| \, d\mu = 0$.

$\square$

15 Fa. 4

4. Consider the following two statements about a function $f\colon [0,1] \to \mathbb{R}$:
(i) $f$ is continuous almost everywhere
(ii) $f$ is equal to a continuous function $g$ almost everywhere.
Does (i) imply (ii)? Prove or give a counterexample. Does (ii) imply (i)? Prove or give a counterexample.

(ii) $\not\Rightarrow$ (i) Consider $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ and $g = 0$.

Then $f \neq g$ only on $\mathbb{Q}$, which has measure 0. So $f = g$ a.e.
But $f$ is continuous nowhere, since for any $x \in [0,1]$, there are sequences $x_n \to x$ s.t. $\{x_n\} \subset \mathbb{Q}$ or $\{x_n\} \subset [0,1] \setminus \mathbb{Q}$.

(i) $\not\Rightarrow$ (ii) Consider $f(x) = \begin{cases} 0 & \text{if } x \in [0, 1/2] \\ 1 & \text{if } x \in (1/2, 1] \end{cases}$.

Assume there was some continuous $g$ s.t. $f = g$ a.e.

Consider $x \in [0, 1/2]$ If $g(x) \neq 0$, then $|g(x)| > 0$.
$g$ continuous $\Rightarrow$ $\exists\ \delta$ s.t. $\forall y \in (x-\delta, x+\delta) \cap [0, 1/2]$, $|g(y) - g(x)| < |g(x)|/2$.
$\Rightarrow$ $|g(y)| \neq 0$ $\forall\ y \in (x-\delta, x+\delta) \cap [0, 1/2]$ contradicting $g = f$ a.e.
Since $(x-\delta, x+\delta) \cap [0, 1/2]$ is not a null set.
Thus $g(x) = 0$ $\forall\ x \in [0, 1/2]$

Similarly $g(x) = 1$ $\forall\ x \in (1/2, 1]$ $\Rightarrow$ $g = f$ not continuous $\Rightarrow \Leftarrow$.

Thus there cannot be a continuous $g$ s.t. $f = g$ a.e.