Real, Fall 2015

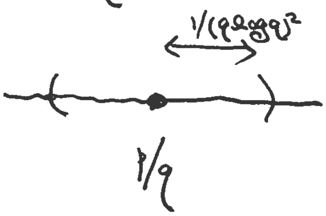
(1) For almost all $x \in [0,1]$ there are at most finitely many rational numbers w/ reduced form p/q s.t. $q \geq 2$ and $\left| x - \frac{p}{q} \right| < \frac{1}{(q \log q)^2}$.

Hint: consider intervals of length $\frac{2}{(q \log q)^2}$ centered at rational points p/q.

For each $x \in [0,1]$, define $Q(x) \in \mathbb{R}^{\geq 0}$ to be the # of $p/q^{\neq 0,1,\, q \geq 2}$ s.t. $|x - p/q| < \frac{1}{(q \log q)^2}$ & let $\mathbb{Q}(x)$ be the set of such p/q. We want to show $\int_0^1 Q(x)\,dx < \infty$, since this implies $Q(x) < \infty$ a.e. we have

$$\int_0^1 Q(x)\,dx = \int_0^1 \sum_{\substack{p/q \in \mathbb{Q} \\ q \geq 2}} \chi_{\mathbb{Q}(x)}\,dx$$

$$\overset{\text{Tonelli}}{=} \sum_{\substack{p/q \in \mathbb{Q} \\ q \geq 2}} \int_0^1 \chi_{\mathbb{Q}(x)}\,dx \leq \sum_{\substack{p/q \in \mathbb{Q} \\ q \geq 2}} \frac{2}{q^2 \log^2 q} \underset{p=1,\dots,q-1}{=} \sum_{q=2}^{\infty} \frac{2(q-1)}{q} \frac{1}{q \log^2 q}$$



$< \infty$ since $\sum_{n=2}^{\infty} \frac{1}{n \log^p n}$ $\begin{cases} \text{converges } p > 1 \\ \text{diverges } p \leq 1 \end{cases}$.

$\square$

(2) f real-valued, nondecreasing on $[0,1]$.

Show there exists a sequence of continuous functions f_n on $[0,1]$ s.t. $f_n \to f$ ptwise.

Index the set of countable discontinuities by $\{x_n\}_{n\in\mathbb{N}}$, where $x_n \equiv 0$ eventually if there are only finitely many discontinuities. Set $a_n \geq 0$ to be the length of the gap at the jump discontinuity x_n, where $a_n \equiv 0$ eventually if there are only finitely many discontinuities.

Define $g = f - \sum_{n\in R\subset\mathbb{N}} a_n \chi_{[x_n,1]} - \sum_{n\in L\subset\mathbb{N}} a_n \chi_{(x_n,1]}$

where $n \in R$ if $f(x_n) = f(x_n+)$ and $n \in L$ if $f(x_n) = f(x_n-)$. Then g is continuous. For simplicity, write $g = f - \sum_{n\in\mathbb{N}} f_n$ (note $\sum_{n\in\mathbb{N}} f_n$ is well-defined, being the sum of positive functions) $\sum_{n\in\mathbb{N}} f_n$ is also real-valued, since $\sum_{n\in\mathbb{N}} f_n(x) \leq \max(f(x+), f(x-)) - f(0)$.

To show g is continuous, we show $g(x-) = g(x+)$ (g is increasing)

If $x \notin \{x_n\}$, it suffices to show $\sum f_n(x+) = \sum f_n(x-)$.

For this X, let $L_X = \{n : X_n < X\}$ and $R_X = \{n : X_n > X\}$. We show $\sum f_n(X-) = \sum f_n(X+) = \sum_{n \in L_X} a_n$. Evaluating $\sum f_n$ on a sequence $y_m \downarrow X$ we find $\sum f_n(X+) \le \sum_{n \in L_X} a_n$ and since $\sum f_n(X) \geq \sum_{n \in L_X} a_n$ we have $\sum f_n(X+) = \sum_{n \in L_X} a_n$. A symmetric procedure shows $\sum f_n(X-) = \sum_{n \in L_X} a_n$. So g is continuous at X.

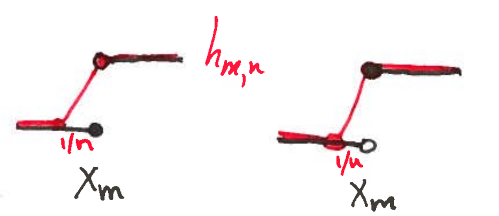
Next, let $X = X_m$. It is clear that

$$g(X-) = f(X-) - \sum_{X_n < X_m} a_n \quad \text{and} \quad g(X+) = f(X+) - \sum_{X_n \le X_m} a_n \, ;$$

since $a_m = f(X+) - f(X-)$, g is continuous at X. So g is continuous.

$$\text{Define } h_{m,n}(x) = \begin{cases} a_m, & X_m < X \le 1 \\ a_m - \dfrac{a_m(X_m - t)}{1/n}, & 0 \le X_m - t \le \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

And define $H_n = \sum_{m=1}^{n} h_{m,n}$.



Finally, define $g_n = g + H_n$.

It is clear that g_n is continuous and $g_n \to f$ pointwise. $\square$

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(3) (X, μ) finite. $f_n \in L^1$ s.t. for each $\varepsilon > 0$ there exists $\delta > 0$ s.t. $\mu(E) < \delta \Rightarrow \int_E |f_n| < \varepsilon$ $(\forall n)$.
Let $f_n \to f$ in measure. Show $f \in L^1$ and $f_n \to f$ in L^1.

<u>Lemma</u> If $f_n \to f$ a.e. and $\int_F |f_n| \leq M \ (\forall n)$, then $\int_F |f| \leq M$.

<u>pf (lemma)</u> $\int_F |f| = \int_F \liminf |f_n| \leq \liminf \int_F |f_n| \leq M$. ✓

Since $f_n \to f$ in measure, there is a subsequence $f_{n_k} \to f$ a.e. Fix $\varepsilon > 0$. By uniform integrability choose $\delta > 0$ s.t. $\int_F |f_{n_k}| \leq \varepsilon$ when $m(F) \leq \delta \ (\forall k)$

By Egoroff's theorem, we may choose E s.t. $f_{n_k} \to f$ uniformly on E and $m(E^c) \leq \delta$. By the lemma, $\int_{E^c} |f| \leq \varepsilon$. Choosing k sufficiently large, we have $\int |f| \leq \int_E |f - f_{n_k}| + \int_{E^c} |f - f_{n_k}| + \int |f_{n_k}|$

$\leq \varepsilon m(X) + \int_{E^c} |f| + \int_{E^c} |f_{n_k}| + \int |f_{n_k}|$

$\leq \varepsilon m(X) + \varepsilon + \varepsilon + \int |f_{n_k}| < \infty$. $\therefore f \in L^1$! ✓

Lastly, we want to show $\lim_{n\to\infty} |f - f_n| = 0$.
Fix $\varepsilon > 0$ and choose $\delta > 0$ s.t. $\int_E |f_n| \leq \varepsilon$ when $m(E) \leq \delta \ (\forall n)$

Choose N s.t. $\forall n \geq N$, $m(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \leq \delta$.

Set $E = \{x : |f_n(x) - f(x)| \geq \varepsilon\}$. Then $\forall n \geq N$,

$$\int |f - f_n| = \int_{\{x : |f_n(x) - f(x)| \leq \varepsilon\}} |f - f_n| + \int_E |f - f_n|$$

$$\leq \varepsilon m(X) + \int_E |f| + \int_E |f_n|$$

$$\leq \varepsilon m(X) + \varepsilon + \varepsilon,$$

where we have once again employed the lemma.

$\therefore \int |f_n - f| \longrightarrow 0$. $\square$

Let $f : [0,1] \to \mathbb{R}$.

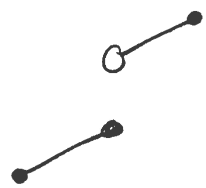
(4) (a) T/F: f continuous a.e. $\Longrightarrow$

$\exists g : [0,1] \to \mathbb{R}$ continuous s.t. $f = g$ a.e.

(b) T/F: $\exists g : [0,1] \to \mathbb{R}$ continuous s.t. $f = g$ a.e. $\Longrightarrow$

f is continuous a.e.

(a) F. Let $f = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ x+1, & \frac{1}{2} < x \le 1. \end{cases}$



Then f is continuous a.e.

Assume there were g as in statement. Since $(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon)$ has positive measure for all $\varepsilon > 0$, we must have $g(x-) = f(x-) = \frac{1}{2}$ and $g(x+) = f(x+) = 3/2$, contradicting g is continuous.

(b) F. Let $f(x) = \begin{cases} 0, & x \in [0,1] - \mathbb{Q} \\ 1, & x \in \mathbb{Q}. \end{cases}$

Then $f = 0$ a.e. But f is continuous nowhere.

$\square$