14 Fa. 1

1. Assume that f is integrable on $(0,1)$. Prove that

$$\lim_{a\to\infty}\int_0^1 f(x)\,x\,\sin(ax^2)\,dx = 0.$$

First let $f = \chi_{(b,c)}$ for $(b,c) \subset (0,1)$.

Then $\lim_{a\to\infty} \int_0^1 \chi_{(b,c)}(x)\, x \sin(ax^2)\, dx$

$= \lim_{a\to\infty} \int_b^c x \sin(ax^2)\, dx$

$= \lim_{a\to\infty} -\frac{1}{2a}\cos(ax^2)\Big]_b^c$

$= \lim_{a\to\infty} \frac{1}{2a}\left[\cos(ab^2) - \cos(ac^2)\right]$

And $\left|\frac{1}{2a}\left[\cos(ab^2) - \cos(ac^2)\right]\right| \le \frac{1}{a} \to 0$ as $a \to \infty$.

$\Rightarrow \lim_{a\to\infty} \int_0^1 \chi_{(b,c)}(x)\, x \sin(ax^2)\, dx = 0$.

By linearity of the integral, this is true when f is a simple function on sets which are finite unions of open intervals.

Since we have the Lebesgue measure, and since $f \in L^1$, for $\varepsilon > 0$, $\exists$ such a simple function ϕ with $\int_0^1 |f - \phi| < \varepsilon$.

Then, $\lim_{a\to\infty} \left| \int_0^1 f(x)\, x\sin(ax^2)\, dx - \int_0^1 \phi(x)\, x\sin(ax^2)\, dx \right|$

$\le \lim_{a\to\infty} \int_0^1 |(f(x) - \phi(x))\, x\sin(ax^2)|$

$\le \lim_{a\to\infty} \int_0^1 |f(x) - \phi(x)|\, dx$ since $|x\sin(ax^2)| \le 1$ on $(0,1)$.

$< \varepsilon$

$\Rightarrow \lim_{a\to\infty} \left| \int_0^1 f(x)\, x\sin(ax^2)\, dx \right| < \varepsilon$, and since ε was abitrary, we conclude

$$\lim_{a\to\infty} \int_0^1 f(x)\, x\sin(ax^2)\, dx = 0.$$

$\square$

14 Fa. 2

2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f and $f_1, f_2, f_3 \ldots$ be real valued measurable functions on X. If $f_n \to f$ in measure and if $F\colon \mathbb{R} \to \mathbb{R}$ is uniformly continuous, prove that $F \circ f_n \to F \circ f$ in measure.

$f_n \to f$ in measure $\iff$ $\mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

NTS $\mu(\{x : |F \circ f_n(x) - F \circ f(x)| \geq \varepsilon\}) \to 0$ as $n \to \infty$.

F uniformly cont. $\iff$ $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in X$, if $y \in (x - \delta, x + \delta)$, then $|F(y) - F(x)| < \varepsilon$.

$\exists N$ s.t. $\forall n \geq N$ $\mu(\{x : |f_n(x) - f(x)| \geq \delta\}) < \varepsilon$.

$$|f_n(x) - f(x)| < \delta \Rightarrow |F \circ f_n(x) - F \circ f(x)| < \varepsilon$$

Thus, $\{x : |f_n(x) - f(x)| < \delta\} \subset \{x : |F \circ f_n(x) - F \circ f(x)| < \varepsilon\}$

$\Rightarrow \{x : |f_n(x) - f(x)| \geq \delta\} \supset \{x : |F \circ f_n(x) - F \circ f(x)| \geq \varepsilon\}$

$$\Rightarrow \mu(\{x : |F \circ f_n(x) - F \circ f(x)| \geq \varepsilon\}) \leq \mu(\{x : |f_n(x) - f(x)| \geq \delta\}) \to 0 \text{ as } n \to \infty.$$

Hence, $F \circ f_n \to F \circ f$ in measure.

$\square$

14 Fa. 3

3. Let f_n be **nonnegative** measurable functions on a measure space $(X, \mathcal{M}, \mu)$ which satisfy $\int f_n \, d\mu = 1$ for all $n = 1, 2, \ldots$. Prove that

$$\limsup_n (f_n(x))^{1/n} \le 1$$

for μ-a.e. x.

$$\lim_{n \to \infty} n^{1/n} = 1$$

$f_n(x) > n$ on a shrinking measure.

14 Fa. 4

4. Let $-\infty < a < b < \infty$. Suppose $F\colon [a,b] \to \mathbb{C}$.

(a) Define what it means for F to be absolutely continuous on $[a,b]$.

(b) Give an example of a function which is uniformly continous but not absolutely continuous. (Remember to justify your answer.)

(c) Prove that if there exists M such that $|F(x) - F(y)| \leq M|x-y|$ for all $x, y \in [a,b]$, then F is absolutely continous. Is the converse true? (Again, remember to justify your answer.)

a)