13 Fa, 1

1. Let μ be a finite Borel measure on $\mathbb{R}$, which is absolutely continuous with respect to the Lebesgue measure m. Prove that $x \mapsto \mu(A+x)$ is continuous for every Borel set $A \subseteq \mathbb{R}$.

Let A be a Borel set and let $c \in \mathbb{R}$.
We need to show $\lim_{x\to c} \mu(A+x) = \mu(A+c)$

If we show $\lim_{x \to c} \mu((A+x) \Delta (A+c)) = 0$,
then

$$\begin{aligned}\lim_{x\to c} |\mu(A+x) - \mu(A+c)| &= \lim_{x\to c} |\mu((A+x)\setminus(A+c)) - \mu((A+c)\setminus(A+x))| \\ &\leq \lim_{x\to c} \mu((A+x)\setminus(A+c)) + \mu((A+c)\setminus(A+x)) \\ &= \lim_{x\to c} \mu((A+x)\Delta(A+c)) \\ &= 0\end{aligned}$$

$\Rightarrow \lim_{x\to c} \mu(A+x) = \mu(A+c)$, finishing the proof.

Thm 3.5 says, since μ finite and $\mu \ll m$, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $m(E) < \delta$ then $\mu(E) < \varepsilon$.
We can find an open $U \supset A$ s.t. $m(U \setminus A) < \delta^*$ $\Rightarrow \mu(U\setminus A) < \varepsilon$.
U open $\Rightarrow U = \bigcup_1^\infty (a_i, b_i)$, disjoint open intervals.
μ finite $\Rightarrow \exists N$ s.t. $\mu(\bigcup_1^N (a_i,b_i)) > \mu(U) - \varepsilon$.
Then $|\mu(A) - \mu(\bigcup_1^N(a_i,b_i))| \leq |\mu(U) - \mu(A)| + |\mu(U) - \mu(\bigcup_1^N(a_i,b_i))|$
$< 2\varepsilon$.

Now, $m(\bigcup_1^N(a_i+x, b_i+x) \Delta \bigcup_1^N(a_i+c, b_i+c)) \leq \sum_1^N m((a_i+x, b_i+x)\Delta(a_i+c, b_i+c))$
$= \sum_1^N 2\min\{|x-c|, (b_i - a_i)\}$
which $\to 0$ as $x \to c$. Thus, we can find a $\delta > 0$ s.t. $\forall x \in (c-\delta, c+\delta)$
$\mu(\bigcup_1^N(a_i+x, b_i+x) \Delta \bigcup_1^N(a_i+c, b_i+c)) < \varepsilon$.

$(A+x \Delta \bigcup_1^N(a_i+x, b_i+x)) = (A \Delta \bigcup_1^N(a_i,b_i))$
$\leq$

$f(x) = \mu(A+x) = \int \chi_{A+x}(y)\, d\mu(y) = \int \frac{d\mu}{dm}(y) \chi_{A+x}(y)\, dm(y)$
$= \int \frac{d\mu}{dm}(y) \chi_A(y-x)\, dm(y)$

* True even if $m(A) = \infty$. Limit of finite case on $[-M, M]$.

$\mu(A+x) = \int_{A+x} d\mu$
$= \int_{A+x} \frac{d\mu}{dm} dm$

$\int \chi_A(y-x) d\mu$

$f(x) = \int \frac{d\mu}{dm} \chi_A(y-x)\, dm(y)$

approx. by step function.

$g(x) = \mu(A+x)$

f = dens. of μ $\qquad f = \frac{d\mu}{dm}$

$$g(x) = \int \chi_A(y) f(y-x)\, dm(y)$$

OR
$$= \int \underbrace{\chi_A(y-x) f(y)}\, dm(y).$$

want to approx by a step func. of $y-x$.

$\Psi(x) = \int \chi_A(y-x)\, dm(y)$ $\qquad$ can approx by step function

13 Fa. 2

2. Let f be a Lebesgue integrable function on $\mathbb{R}$, and assume that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty.$$

Prove that $g(x) = \sum_{n=1}^{\infty} f(a_n x)$ converges almost everywhere and is integrable on $\mathbb{R}$. Also, find an example of a Lebesgue integrable function f on $\mathbb{R}$ such that $g(x) = \sum_{n=1}^{\infty} f(nx)$ converges almost everywhere but is not integrable.

First, we find the example.

Let $f = \chi_{[0,1]}$. $\Rightarrow g(x) = \sum_{n=1}^{\infty} \chi_{[0,1]}(nx)$.

$x \in (1/2, 1] \Rightarrow 2x \notin [0,1] \Rightarrow g(x) = 1$.

$x \in (1/3, 1/2] \Rightarrow 3x \notin [0,1] \Rightarrow g(x) = 2$.

This continues and we see that $g(x) = \lfloor \frac{1}{x} \rfloor$ on $(0,1]$.

And $g(x) = 0 \ \forall x \notin [0,1]$. Thus $g(x)$ converges almost everywhere.

But $\int g(x) = \int_0^1 \lfloor \frac{1}{x} \rfloor = \int_{1/2}^1 1 + \int_{1/3}^{1/2} 2 + \int_{1/4}^{1/3} 3 + \cdots$

$= 1/2 + 1/3 + 1/4 + \cdots$

$= \infty$. $\square$

Now, we prove the result.

Assume g diverges on a non-null set, E

Then consider $\{a_n x : n \in \mathbb{N}, x \in E\}$, call this A.

Claim: $\int_A f = \int_E g = \infty$.

$\int_E g = \int_E \sum_1^\infty f(a_n x)$

$\int g(x) = \int \sum f(a_n x)\, dx$

$\int f(a_n x)\, dm = \frac{1}{a_n} \int f(x)\, dm$

$u = a_n x$ $\qquad \int \frac{1}{a_n} f(u)\, du$

$du = a_n\, dx$

13Fa. 3

3. Assume $b > 0$. Show that the Lebesgue integral

$$\int_1^\infty x^{-b} e^{\sin x} \sin(2x)\, dx$$

exists if and only if $b > 1$.

?

$$\left| \int_1^\infty x^{-b} e^{\sin x} \sin(2x)\, dx \right| \le \int_1^\infty |x^{-b} e|\, dx$$
$$\le M$$

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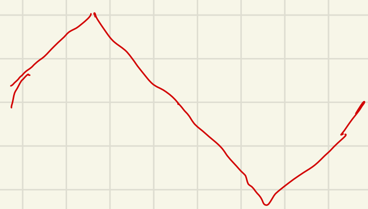
Find $\pm\infty$ for $b \le 1$.

sin(2x)

Break down into periods of $\sin x$.

$e^{\sin x}$



Section by section.

$\sum M n^{-b} = \infty$

13 Fa. 4

4. Suppose that F is the distribution function of a Borel measure μ on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$. Prove that

$$\int_{-\infty}^{\infty} (F(x+a) - F(x))\, dx = a$$

for all $a > 0$.

$$\begin{aligned}
\int_{-\infty}^{\infty} (F(x+a) - F(x))\, dx &= \int_{-\infty}^{\infty} \mu((x, x+a])\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(x,x+a]}(y)\, d\mu(y)\, dm(x) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{(x,x+a]}(y)\, dm(x)\, d\mu(y) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \chi_{[y-a,y)}(x)\, dm(x)\, d\mu(y) \\
&= \int_{-\infty}^{\infty} (y - (y-a))\, d\mu(y) \\
&= \int_{-\infty}^{\infty} a\, d\mu(y) = a\,\mu(\mathbb{R}) = a.
\end{aligned}$$

Because $y \in (x, x+a]$

$\iff x < y$ AND $y \le x+a$

$\iff x < y$ AND $y-a \le x$

$\iff x \in [y-a, y)$.

Because $\chi_{(x,x+a]}(y) \in L^+(\mathbb{R} \times \mathbb{R})$ and μ, m are σ-finite, so we can use Tonelli to swap iterated integrals.