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Real Analysis Exam Fall 2012

Problem 1. Let m be the Lebesgue measure on $X = [0,1]$. If

$$m(\limsup_{n\to\infty} A_n) = 1, \quad m(\liminf_{n\to\infty} B_n) = 1,$$

prove that $m\left(\limsup_{n\to\infty}(A_n \cap B_n)\right) = 1$, where

$$\limsup_{n\to\infty} A_n = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k, \quad \liminf_{n\to\infty} B_n = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} B_k.$$

Solution. Let $A = \limsup A_n$ and $B = \limsup B_n$. Since

$$1 = m(\liminf B_n) \le m(\limsup B_n) \le m(X) = 1 \implies m(B) = 1.$$

Futhermore, since $m(A) = m(B) = m(X) = 1$, $m(A^c) = m(B^c) = 0$.

Now,

$$\limsup A_n \cap B_n = \bigcap n = 1^{\infty} \bigcup_{k=n}^{\infty} (A_k \cap B_k) = \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \cap \bigcup_{k=n}^{\infty} B_k \right) = \limsup A_n \bigcap \limsup B_n.$$

Finally,

$$1 = m(A \cup B) = m(A) + m(B) - m(A \cap B) = 2 - m(A \cap B) \implies m(A \cap B) = 1$$

since $m(A \cup B) \ge m(A) = 1$.

Thus,

$$m(A \cap B) = m\left(\limsup_{n\to\infty}(A_n \cap B_n)\right) = 1.$$

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Problem 2. Assume that $f : X \to [0, \infty)$ is measurable. Find

$$\lim_n \int_X n \log\left[1 + \frac{f(x)}{n}\right] d\mu.$$

Solution.

$$(n+1)\log\left(1+\frac{f}{n+1}\right) - n\log\left(1+\frac{f}{n}\right) = n\log\left(\frac{1+\frac{f}{n+1}}{1+\frac{f}{n}}\right) - \log\left(1+\frac{f}{n+1}\right) \leq 0 \quad \text{for all } n$$

since

$$\frac{1+\frac{f}{n+1}}{1+\frac{f}{n}} \leq 1 \qquad \text{and} \qquad 1+\frac{f}{n+1} \geq 1.$$

Thus, we have that $f_n = n\log\left(1+\frac{f}{n}\right)$ is decreasing in terms of n so $f_{n+1} \leq f_n$. Now, because f is measurable, f_n is measurable since the natural log is continuous.

$$\begin{aligned}
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{\log\left(1+\frac{f(x)}{n}\right)}{\frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{\frac{1}{1+\frac{f(x)}{n}} \frac{-f(x)}{n^2}}{\frac{-1}{n^2}} \qquad \text{L'Hopital's Rule} \\
&= \lim_{n\to\infty} \frac{f(x)}{1+\frac{f(x)}{n}} \\
&= f(x) \qquad \text{for all } x.
\end{aligned}$$

Now, let $g_n(x) = f(x) - f_n(x)$. Then

$$f(x) \geq \log(1+f(x)) = f_1(x)$$

for all $f(x) \geq 0$ and since $f_1(x) \geq f_n(x)$ for all $n \geq 1$, we have that $g_n(x) \geq 0$ for all x. We would like to use the Monotone Convergence Theorem on g_n.

1. $\{g_n\} \in L^+$ since it is measurable and positive.
2. $f_n \geq f_{n+1}$ so $g_n \leq g_{n+1}$ for all n.
3. $g_n \to 0$ for all x since $f_n \to f(x)$.

Thus, by MCT,

$$\lim_n \int_X n\log\left[1+\frac{f(x)}{n}\right] d\mu = \int_X f(x)dx.$$

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Problem 3. Let $f \in L^1(m)$. For $k = 1, 2, ...$ let f_k be the step function defined by

$$f_k(x) = k\int_{j/k}^{(j+1)/k} f(t)dt$$

$$\text{for } \frac{j}{k} < x \leq \frac{j+1}{k}, j = 0, \pm 1, ...$$

Show that f_k converge to f in L^1 as $k \to \infty$.

Solution. First, since $f \in L^1$ $f \in L^1_{loc}$. For each x, there exists some j such that $x \in (\frac{j}{k}, \frac{j+1}{k}]$ and so

$$f_k(x) = k\int_{j/k}^{(j+1)/k} f(t)dt = \frac{1}{m((j/k, (j+1)/k])}\int_{(j/k,(j+1)/k])} f(t)dt.$$

Since $(j/k, (j+1)/k]$ shrinks nicely to x as $k \to \infty$, by the Lebesgue Differentiation Theorem

$$\lim_{k\to\infty} f_k(x) = f(x) \qquad \text{a.e..}$$

Now, we would like to use Dominated Convergence Theorem.

1. Let $g_k(x) = |f(x) - f_k(x)|$. Then, let $\int |f(x)| = M < \infty$ since $f \in L^1$. Then

$$\begin{aligned}
\int |g_k(x)|dx &= \int |f(x) - f_k(x)|dx \\
&\leq \int |f(x)|dx + \int |f_k(x)|dx \\
&= M + \sum_j k\int_{j/k}^{(j+1)/k} \left|\int_{j/k}^{(j+1)/k} f(x)dx\right| dx \\
&= M + \sum_j k\, m\left((\frac{j}{k}, \frac{j+1}{k})\right)\left|\int_{j/k}^{(j+1)/k} f(x)dx\right| \qquad \left|\int_{j/k}^{(j+1)/k} f(x)dx\right| \text{ is constant} \\
&= M + \sum_j k\frac{1}{k}\left|\int_{j/k}^{(j+1)/k} f(x)dx\right| \\
&\leq M + \sum_j \int_{j/k}^{(j+1)/k} |f(x)|dx \\
&= M + M = 2m
\end{aligned}$$

Thus, $g_k(x) \in L^1$ for all k.

2. $g_k(x) \to 0$ a.e. since $f_k \to f$ a.e.

3. For each $x > 0$ and k there exists some $j \geq 0$ such that $x \in (j/k, (j+1)/k]$. Then $x \in (0, j+1]$. Similarly, if $x \leq 0$, then $x \in (j, 0]$. Thus,

$$g_k(x) \leq |f(x)| + |f_k(x)| \leq |f(x)| + \begin{cases} M\chi_{(0,j+1]}(x) \\ M\chi_{(j,0]}(x) \end{cases} \in L^1.$$

Thus, by the dominated convergence theorem,

$$\lim_{k\to\infty}\int |f - f_k|dx = \lim_{k\to\infty} g_k(x)dx = \int \lim_{k\to\infty} g_k(x)dx = 0.$$

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Problem 4 (Folland, 3.4.25, p.100)**.** If E is Borel set in $\mathbb{R}^n$ the density $D_E(x)$ of E at x is defined as
$$D_E(x) = \lim_{r\to 0} \frac{m(E\cap B(x,r))}{m(B(x,r))},$$
whenever the limit exists [Here m denotes the Lebesgue measure and $B(x,r)$ is the open ball with center at x and radius r.]

(a) Show that $D_E(x) = 0$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.

(b) For $\alpha \in (0,1)$ find an example of E and x such that $D_E(x) = \alpha$.

(c) Find an example of E and x such that $D_E(x)$ does not exist.

Solution.

(a)
$$m(E\cap B(x,r)) = \int_{B(x,r)} \chi_E(x)dx$$

Since $\chi_E \in L^1_{loc}$ because E is Borel measurable, and certainly $B(x,r)$ shrinks nicely to x, so by the Lebesgue Dominated Convergence Theorem
$$\lim_{r\to 0}\frac{m(E\cap B(x,r))}{m(B(x,r))} = \lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)}\chi_E(x)dx = \chi_E(x) \qquad \text{a.e.}$$

Thus, $D_E(x) = 0$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.

(b) We will work in $\mathbb{R}^2$. Fix $\alpha \in (0,1)$. For $x = 0$ let $\theta = 2\phi\alpha$. Let E be the set of points in a θ sector of x. Specifically,
$$E = \{(R\cos\gamma, R\sin\gamma)\,|\, R \geq 0, 0 \leq \gamma \leq \theta\}.$$

Then
$$B(0,r)\cap E = \{(R\cos\gamma, R\sin\gamma)\,|\, r > R \geq 0, 0 \leq \gamma \leq \theta\}.$$

Thus,
$$m(B(0,r)\cap E) = \frac{\theta}{2\pi}m(B(0,r) = \alpha m(B(0,r))$$

so
$$\lim_{r\to 0}\frac{m(E\cap B(0,r))}{m(B(0,r))} = \lim_{r\to 0}\alpha = \alpha.$$

(c) We will work in $\mathbb{R}^1$. Let $B_n(-\frac{1}{n}, \frac{1}{n})$. Fix $x = 0$.
Let $E = \bigcap_{n=1}^{\infty}(B_{(2n-1)!}\backslash B_{(2n)!})$.

If n is odd then we note that $B_{n!} \backslash B_{(n+1)!} \subset E$ so

$$\frac{m(E \cap B_{n!})}{m(B_{n!})} \geq \frac{m(B_{n!} \backslash B_{(n+1)!})}{m(B_{n!})} = \frac{\frac{2}{n!} - \frac{2}{(n+1)!}}{\frac{2}{n!}} = 1 - \frac{1}{n} \to 1.$$

Thus $D_E(0) \geq 1$.

If n is even, then $E \cap B_{n!} \subset B_{(n+1)!}$ so

$$\frac{m(E \cap B_{n!})}{m(B_{n!})} \leq \frac{m(B_{(n+1)!})}{m(B_{n!})} = \frac{\frac{2}{(n+1)!}}{\frac{2}{n!}} = \frac{1}{n+1} \to 0.$$

So $D_E(0) = 0$. Thus, since there are infinitely many even and odd n, $D_E(0)$ does not exist.

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