Fall 2012:

(1.) m Lebesgue on $X=[0,1]$. Suppose $m(\limsup_{n\to\infty} A_n) = 1$ and $m(\liminf_{n\to\infty} B_n) = 1$. Show that $m(\limsup_{n\to\infty}(A_n \cap B_n)) = 1$

Recall that $\limsup_{n\to\infty} E_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} E_n = \{x : x\in E_n \text{ for infinitely many } n\}$

$\liminf_{n\to\infty} E_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} E_n = \{x : x\in E_n \text{ for all but finitely many } n\}$.

So: $x \in \limsup_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n$

$\Rightarrow x\in \limsup_{n\to\infty} A_n$ and $x \in \liminf_{n\to\infty} B_n$

$\Rightarrow$ $x\in A_n$ for infinitely many n

$x \in B_n$ for all but finitely many n

$\Rightarrow \exists$ seq. n_j s.t. $x\in A_{n_j}$ for all $j=0,1,2\ldots$

$\exists N$ s.t. $x\in B_n$ for all $n\geq N$

$\Rightarrow x\in A_{n_j}\cap B_{n_j}$ for all $n_j \geq N$, hence $x\in A_n\cap B_n$ for infinitely many n

$\Rightarrow x\in \limsup_{n\to\infty} A_n\cap B_n$.

So $m(\limsup_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n) \leq m(\limsup_{n\to\infty} A_n\cap B_n)$.

But $m(\limsup_{n\to\infty} A_n \cap \liminf_{n\to\infty} B_n] = m(\limsup A_n) + m(\liminf B_n)$

$- m(\limsup A_n \cup \liminf B_n)$

$= 1+1-\mu(\limsup A_n \cup \liminf B_n)$

$\geq 2-1 = 1$

Hence $\underline{m(\limsup A_n \cap \liminf B_n) = 1}$

(2.) $f: X \to [0,\infty)$ measurable.

Find $\displaystyle\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu$.

Consider the sequence of fns $n\log\left(1+\frac{f(x)}{n}\right)$

Then consider $\displaystyle\frac{d}{dn}\left(n\log\left(1+\frac{f(x)}{n}\right)\right) = \log\left(1+\frac{f(x)}{n}\right) + n\left(\frac{-f(x)n^{-2}}{1+\frac{f(x)}{n}}\right)$

$\displaystyle= \log\left(1+\frac{f(x)}{n}\right) - \frac{f(x)}{n+f(x)} = \log\left(1+\frac{f(x)}{n}\right) - \frac{\frac{f(x)}{n}}{1+\frac{f(x)}{n}} \quad (*)$

$\Rightarrow$ Now consider the following for $y \geq 0$:

for $y=0$, $\frac{y}{y+1} = 0 = \log(1+y)$; and:

$\displaystyle\frac{d}{dy}\left(\frac{y}{y+1}\right) = \frac{-y}{(1+y)^2} + \frac{1}{(1+y)} = \frac{1}{(1+y)^2}$

$\displaystyle\frac{d}{dy}\left(\log(1+y)\right) = \frac{1}{1+y}$;

the two functions start in the same place, but one grows faster than the other $\forall y$

Hence for $y \geq 0$, $\displaystyle\frac{d}{dy}\left(\frac{y}{y+1}\right) \leq \frac{d}{dy}\left(\log(1+y)\right)$, hence

$\displaystyle\frac{y}{y+1} \leq \log(1+y)$ for $y \in [0,\infty)$

Hence since $\frac{f(x)}{n} \geq 0$, we have that $\displaystyle\frac{\frac{f(x)}{n}}{1+\frac{f(x)}{n}} \leq \log\left(1+\frac{f(x)}{n}\right)$,

hence $(*)$ is positive for all x, n, hence the sequence $f_n(x)$ is increasing as $n \to \infty$ for all x.

Therefore we may apply MCT:

$$\lim_{n\to\infty} \int n\log\left(1+\frac{f(x)}{n}\right) dx = \int \lim_{n\to\infty} n\log\left(1+\frac{f(x)}{n}\right) dx \quad (**)$$

Now see that, by L'Hôpital's rule:

$$\lim_{n\to\infty} n\log\left(1+\frac{f(x)}{n}\right) = \lim_{n\to\infty} \frac{\log\left(1+\frac{f(x)}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{f(x)}{n}}\right)\left(\frac{-f(x)}{n^2}\right)}{\frac{-1}{n^2}}$$

$\displaystyle= \lim_{n\to\infty} \frac{f(x)}{1+\frac{f(x)}{n}} = f(x)$, hence:

$$(**) = \int f(x)\, dx$$

(3) $f \in L^1(m)$. Let $f_k(x) = k\int_{j/k}^{j+1/k} f(t)\,dt$ for $x \in \left[\frac{j}{k}, \frac{j+1}{k}\right] = I_k^j$

Show that $f_k \to f$ in L^1.

See that $f_k(x) = \sum_j k \chi_{I_j^k}(x) \int_{I_j^k} f(t)\,dt$

→ Now suppose that $f(x) = \sum_{i=1}^n a_i \chi_{E_i}(t)$ (for E_i = finite union of intervals, an integral simple function)

Then $\|f - f_k\|_{L^1} = \int \left| \sum_{i=1}^n a_i \chi_{E_i}(x) - \sum_{j=\infty}^{\infty} k \chi_{I_j^k}(x) \left(\int_{I_j^k} \sum_{i=1}^n a_i \chi_{E_i}(t) \right) \right| dx$

Let j_x be s.t. $x \in I_{j_x}^k$ and let k be large enough s.t. $x \in I_{j_x}^k \subseteq E_{i_x}$ some i_x

$$= \int \left| a_{i_x} - k\left(\int_{I_{j_x}^k} a_{i_x}\,dt \right) \right| dx$$

$m(I_{j_x}^k) = \frac{1}{k}$

$$= \int \left| a_{i_x} - k a_{i_x} m(I_{j_x}^k) \right| dx = \int |a_{j_x} - a_{j_x}|\,dx = 0$$

→ See that:

$$\int_{\mathbb{R}} |f_k(x)| = \int_{\mathbb{R}} \left| \sum_j k \chi_{I_j^k}(x) \left(\int_{I_j^k} f(t)\,dt \right) \right| dx$$

$$\leq \int_{\mathbb{R}} k \sum_j \chi_{I_j^k}(x) \left(\int_{I_j^k} |f(t)|\,dt \right) dx$$

$$= k \sum_j \int_{\mathbb{R}} \int_{I_j^k} \chi_{I_j^k}(x) |f(t)|\,dt\,dx$$

$$= k \sum_j \int_{I_j^k} \int_{\mathbb{R}} \chi_{I_j^k}(x) |f(t)|\,dx\,dt$$

$$= k \sum_j \int_{I_j^k} \frac{1}{k} |f(t)|\,dt = \sum_j \int_{I_j^k} |f(t)|\,dt = \int_{\mathbb{R}} |f(t)|\,dt < \infty,$$

hence $f_k \in L^1(m)$ for all k.

→ Since f, f_k are L^1, we know that for $\varepsilon > 0$, $\exists$ int. simple fns ϕ, ϕ_k s.t. $\|f - \phi\| < \varepsilon$ and $\|f_k - \phi_k\| < \varepsilon$, hence for $\varepsilon > 0$,

$$\|f - f_k\|_{L^1} = \int |f - f_k| = \int |f - \phi + \phi + \phi_k - \phi_k - f_k| \leq \int (|f - \phi| + |f_k - \phi_k| + |\phi - \phi_k|)\,dx$$

$< \varepsilon + \varepsilon + \int |\phi - \phi_k|\,dx = \varepsilon + \varepsilon + 0 = 2\varepsilon$, hence $f_k \to f$ in L^1

by above, for k large enough

(4) E Borel in $\mathbb{R}^n$. Let $D_E(x) = \lim_{r\to 0} \frac{m(E\cap B_r(x))}{m(B_r(x))}$

(a) Show $D_E(x) = 1$ for a.e. $x \in E$
$D_E(x) = 0$ for a.e. $x \notin E$.

(b) Find E, x s.t. $D_E(x) = \alpha \in (0,1)$

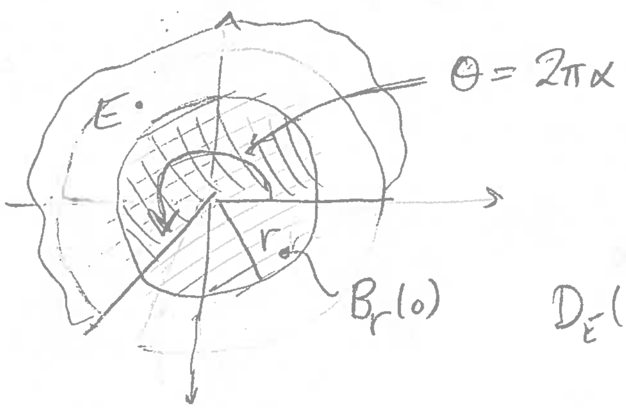
(c) Find E, x s.t. $D_E(x)$ DNE

(a) Define the measure $d\nu = \chi_E \, dm$

Then: $D_E(x) = \lim_{r\to 0} \frac{m(E\cap B_r(x))}{m(B_r(x))} = \lim_{r\to 0} \frac{\int_{B_r(x)} \chi_E}{m(B_r(x))} = \lim_{r\to 0} \frac{\nu(B_r(x))}{m(B_r(x))}$

$= \frac{d\nu}{dm} = \chi_E$ almost everywhere; hence $D_E(x) = 1$ for a.e. $x \in E$, and $= 0$ for a.e. $x \in E^c$

(b) Let E be the region pictured:



See that the area of $E\cap B_r(0)$ is $\frac{1}{2}\theta r^2 = \frac{1}{2} 2\pi\alpha r^2$
$= \alpha\pi r^2$

Therefore:

$D_E(0) = \lim_{r\to 0} \frac{m(E\cap B_r(x))}{m(B_r(x))} = \frac{\alpha \cancel{\pi r^2}}{\cancel{\pi r^2}} = \alpha$

(c) Consider $\mathbb{R}$ and $E = \bigcup_{n \text{ odd}} \left(\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$

→ $\underline{m \text{ odd}}$: Let $r = \frac{1}{2^{m-1}}$

?

Then $m(B_r(0)\cap E) = \sum_{\substack{n\ge m \\ n \text{ odd}}} \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) = \frac{4}{3}\cdot\frac{1}{2^m}$

and $m(B_r(0)) = \frac{2}{2^{m-1}}$, hence $\frac{m(B_r(0)\cap E)}{m(B_r(0))} = \frac{2^{m-1}\cdot 4}{2\cdot 2^m \cdot 3} = \frac{2^{m+1}}{2^{m+1}3} = \frac{1}{3}$

→ $\underline{m \text{ even}}$: Let $r = \frac{1}{2^{m-1}}$ (E is empty from $r = \frac{1}{2^{m+1}}$ to $r = \frac{1}{2}$)

Then $m(B_r(0)\cap E) = m(B_{\frac{1}{2}r}(0)\cap E) = \sum_{\substack{n\ge m+1 \\ n \text{ odd}}} \left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right) = \frac{4}{3}\cdot\frac{1}{2^{m+1}}$

and $m(B_r(0)) = \frac{2}{2^{m-1}}$, so $\frac{m(B_r(0)\cap E)}{m(B_r(0))} = \frac{4\cdot 2^{m-1}}{3\cdot 2^{m+1}\cdot 2} = \frac{2^{m+1}}{3\cdot 2^{m+1}\cdot 2} = \frac{1}{6}$

→ Hence limit DNE since alt. for n even/o