Real, Fall 2012

---

(1) m Lebesgue on $[0,1]$.

If $m(\limsup_n A_n) = 1$ and $m(\liminf_n B_n) = 1$,

show $m(\limsup_n (A_n \cap B_n)) = 1$.

Recall $\limsup_n A_n = \bigcap_K \bigcup_{n \geq K} A_n \ni x$

iff x is in infinitely many A_n. And

$\liminf_n B_n = \bigcup_K \bigcap_{n \geq k} B_n \ni x$ iff x is in

all but finitely many B_n.

We want to use

$$\underbrace{m\left(\limsup_n A_n\right)}_{1} + \underbrace{m\left(\liminf_n B_n\right)}_{1} = \underbrace{m\left(\limsup_n A_n \cup \liminf_n B_n\right)}_{\geq 1 \text{ and } \leq 1 = \text{measure } [0,1] \;\Rightarrow\; = 1}$$

$$+ \, m\left(\limsup_n A_n \cap \liminf_n B_n\right).$$

So we hope $m\left(\limsup_n (A_n \cap B_n)\right) \geq$ $(*)$

$$m\left(\limsup_n A_n \cap \liminf_n B_n\right) = 2 - 1 = 1.$$

Indeed, $x \in \limsup_n A_n \cap \liminf_n B_n$

$\Longrightarrow$ $x \in A_n$ for infinitely many n
& $x \in B_n$ for all but finitely many n

$\Longrightarrow$ $x \in A_n \cap B_n$ for infinitely many n

$\Longrightarrow$ $x \in \limsup_n A_n \cap B_n$

$$\therefore \limsup_n A_n \cap \liminf_n B_n \subset \limsup_n A_n \cap B_n$$

and we're done since $(*)$ follows. $\square$

(2) $f : X \to [0, \infty)$.

Evaluate $\lim_{n\to\infty} \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu$.

Since $\log\left(1 + \frac{f(x)}{n}\right) \geq 0$

we want to use the MCT.

We show first that $\lim_{n\to\infty} n \log\left(1 + \frac{f(x)}{n}\right)$

converges to $f(x)$ everywhere.

If $f(x) = 0$, this is clearly true.

Let $f(x) \neq 0$. Then $n \log\left(1 + \frac{f(x)}{n}\right)$

$$= f(x) \frac{\log\left(1 + \frac{f(x)}{n}\right)}{\frac{f(x)}{n}} \xrightarrow[n\to\infty]{} f(x) \frac{d}{dx}\Big|_{x=1} \log x$$

$= f(x) \cdot 1$ as claimed.

Next we need to show

$n \log\left(1 + \frac{f(x)}{n}\right)$ increases w/ n for all X.

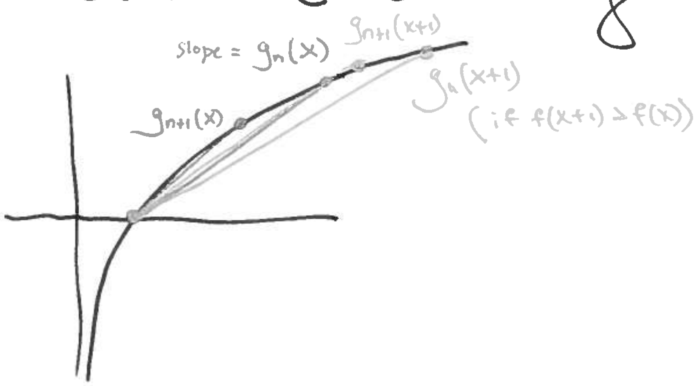
$\| f(x) \neq 0$

(again, we need only consider where $f(x) \neq 0$ since trivially true otherwise.)

$$f(x) \frac{\log\left(1 + \frac{f(x)}{n}\right)}{\frac{f(x)}{n}} \equiv f(x)\, g_n(x).$$

To show $g_n(x)$ increases, since log is increasing it suffices to show the second derivative of log is negative at $X=1$

which is of course true.



Clearly, the slopes $g_n(X)$ increase as $\frac{f(x)}{n}$ decreases, i.e. as n increases.

This proves the $g_n(X)$ are increasing $\forall x$. (w/n)

$$\therefore \text{By the MCT, } \lim_{n\to\infty} \int_X n \log\left(1 + \frac{f(x)}{n}\right) d\mu = \int_X f(x)\, d\mu. \quad \square$$

(3) $f \in L^1(m)$. For $k = 1, 2, \ldots$

set $f_k(x) = k \int_{j/k}^{(j+1)/k} f$ (w.d. since $f \in L^1$)

where $x \in \left(\frac{j}{k}, \frac{j+1}{k} \right]$, $j = 0, \pm 1, \ldots$.

Show $f_k \to f$ in L^1.

First we show the result for $f = \chi_{[a,b]}$. Fix k.

For those j s.t. $\left(\frac{j}{k}, \frac{j+1}{k} \right] \subset [a,b]$ we

have $f_k(x) = k \int_{j/k}^{(j+1)/k} \chi_{[a,b]} = 1$ $\left(x \in \left(\frac{j}{k}, \frac{j+1}{k}\right]\right)$.

Hence for such x, $|f(x) - f_k(x)| = 0$. Examining

what happens at the endpoints, let $a \in \left(\frac{j}{k}, \frac{j+1}{k} \right]$. For x in this interval,

Then $|f_k(x)| \leq k \int_{j/k}^{(j+1)/k} 1 \leq 1$. So $\int_{j/k}^{(j+1)/k} |f - f_k| \leq \frac{2}{k}$ and similarly

for the segment containing b. So $\int |f - f_k| \leq \frac{4}{k} \to 0$ as $k \to \infty$. ✓

Next we show the result for $f = \chi_{\bigcup [a_j, b_j]}$

$[a_j, b_j]$ disjoint, except possibly at endpoints.

Since we are only interested in $f \in L^1$, we demand

$\sum_j (b_j - a_j) < \infty$. Fix $\varepsilon > 0$.

Observe $f = \sum_j \underset{\underset{f^j}{|||}}{\chi_{[a_j, b_j]}}$ and $(f)_k = \underbrace{\sum_j (f^j)_k}_{\substack{\text{disjoint} \\ \text{supports} \\ \text{w/ } j}}$.

As previously, $|(f^i)_k| \le 1$, so for large n

$\sum_{i \geq n} b_j - a_j \leq \varepsilon$ and $\sum_{i \geq n} \int_{a_i}^{b_i} |f - f_k| \leq 2\varepsilon$.

On the remaining finitely many intervals we may take n large enough s.t. $\int_{a_i}^{b_i} |f - f_k| \le \varepsilon$

by the result proved previously. So $\int |f - f_k| \to 0$.

Now we show if f & $\phi \in L^1$, $\int |\phi - \phi_k| \longrightarrow 0$ and $\int |f - \phi| \le \varepsilon$, then

$\int |f - f_k| \sim \varepsilon$ for $k \gg 1$.

$$\int |f - f_k| \le \underbrace{\int |f - \phi|}_{\le \varepsilon} + \underbrace{\int |\phi - \phi_k|}_{\le \ \varepsilon \text{ for } k \gg 1} + \underbrace{\int |\phi_k - f_k|}_{\substack{\text{remains to} \\ \text{estimate.} \\ \le \varepsilon}}$$

$$\int |\phi_k - f_k| \le \sum_j \int_{j/k}^{(j+1)/k} k \underbrace{\int_{j/k}^{(j+1)/k} |\phi - f|}_{\substack{\text{just a} \\ \text{constant for every } j}}$$

$$= \sum_j \int_{j/k}^{(j+1)/k} |\phi - f| \underbrace{\int_{j/k}^{(j+1)/k} k}_{= 1}$$

$$= \sum_j \int_{j/k}^{(j+1)/k} |\phi - f| = \int |\phi - f| \le \varepsilon.$$

$\therefore \int |f - f_k| \sim \varepsilon$ as $k \to \infty$. ✓

Let E be Borel, we show the result is true for $f = \chi_E$. Indeed, let $\underset{E}{\underset{\cup}{U}} = \bigcup (a_j, b_j)$ disjoint be s.t. $m(U - E) = \int |\chi_U - \chi_E| \le \varepsilon$. By the previous two results, $\int |f - f_k| \to 0$. ✓

It follows by the triangle inequality that the result holds for simple functions.

Finally, let $f \in L^1$ and let ϕ_n be simple s.t. $\int |f - \phi_n| \leq \frac{1}{n}$.
Then by what we proved earlier, since the result is true for the ϕ_n it is true for f, and we are done. $\square$

---

(4) $E \subset \mathbb{R}^n$ Borel

Define $D_E(x) = \lim_{r \to 0} \dfrac{m(E \cap B(x,r))}{m(B(x,r))}$.

(a) Show $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \notin E$.

Define the Borel measure λ by $\lambda(F) = m(E \cap F)$
Then clearly $\lambda \ll m$ and λ is σ-finite.
Write $\lambda(F) = \int_F f \, dm$.

Then by the main theorem,

$$\lim_{r\to 0} \frac{\lambda(B(x,r))}{m(B(x,r))} = f(x) \text{ a.e.}$$

Moreover, setting $g = 1$ on E and $g=0$ on E^c

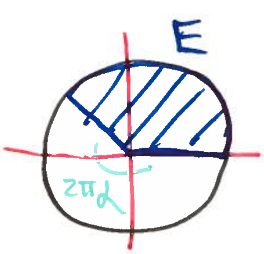
we have $\displaystyle\int_F g\,dm = \int_{F\cap E} dm + \cancelto{0}{\int_{F\cap E^c} g\,dm} = m(F\cap E) = \lambda(F).$

So $f=g$ a.e., hence (a) is proved. $\square$

(b) For $\alpha \in (0,1)$ find E ~~&~~ X s.t.

$D_E(x) = 1.$

$E \subset \mathbb{R}^2$, $E = \left\{ re^{2\pi i\theta} : \begin{array}{c} 0\le r\le 1 \\ 0\le\theta\le\alpha \end{array} \right\}$ and $x=0$.



Clearly, $\dfrac{m(E\cap B(0,r))}{m(B(0,r))} = \dfrac{\alpha\pi}{\pi} = \alpha$

$\Rightarrow D_E(x) = \alpha.$ $\square$

(c) Show E & x s.t. $D_E(x)$ does not exist

$$E = \bigcup_{\substack{n\geq 1\\ \text{odd}}}^{\infty} \left(2^{-(n+1)}, 2^{-n}\right).$$

For n odd: $$m\left(E \cap [-2^{-n}, 2^{-n}]\right) = \sum_{\substack{m\geq n\\ m \text{ odd}}}^{\infty} 2^{-m} - 2^{-(m+1)}$$

$$= -\sum_{m\geq n}^{\infty} \underbrace{(-1)^m 2^{-m}}_{=\left(-\frac{1}{2}\right)^m} = -\left(\underbrace{\frac{1}{1-\left(-\frac{1}{2}\right)}}_{=3/2} - \sum_{m=0}^{n-1} \left(-\tfrac{1}{2}\right)^m\right)$$

$$= \frac{1-\left(-\frac{1}{2}\right)^n}{3/2} - \frac{1}{3/2} = \frac{2}{3}\left(+\frac{(-1)^{n+1}}{2^n}\right) = \frac{2}{3}\frac{1}{2^n}.$$

$$\Longrightarrow \frac{m\left(E\cap[-2^{-n}, 2^{-n}]\right)}{m\left([-2^{-n}, 2^{-n}]\right)} = \frac{\frac{2}{3}\frac{1}{2^n}}{2^{-n+1}} = \frac{1}{3}.$$

For n even: $\left(2^{-(n+1)}, 2^{-n}\right)$ does not appear in E.

So $$m\left(E\cap[-2^{-n}, 2^{-n}]\right) = \sum_{\substack{m\geq n+1\\ m \text{ odd}}} 2^{-m} - 2^{-(m+1)} \underset{\text{by above}}{=} \frac{2}{3}\frac{1}{2^{n+1}}.$$

And $$\frac{m\left(E\cap[-2^{-n}, 2^{-n}]\right)}{m\left([-2^{-n}, 2^{-n}]\right)} = \frac{\frac{2}{3}\frac{1}{2^{n+1}}}{2^{-n+1}} = 1/6.$$ So $D_E(x)$ does not exist. $\square$