Fall 2011:

(1.) $f \geq 0$; suppose $f \in L^1([0,\infty))$. Find $\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx$

See that $\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx = \lim_{n\to\infty} \int_0^\infty \chi_{[0,n]}(x)\,\frac{1}{n}\,x\,f(x)\,dx$

So let $|f_n(x)| = \left|\frac{1}{n}\underbrace{\chi_{[0,n]}(x)\,x}\,f(x)\right|$

$\leq \left|\frac{1}{n}\cdot n\cdot f(x)\right| = |f(x)| = f(x) \in L^1$ (non-neg)

Therefore $|f_n(x)|$ is bdd by an L^1 fn. for all n, so apply DCT:

$$\lim_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx = \lim_{n\to\infty}\int_0^\infty \chi_{[0,n]}(x)\,\frac{1}{n}\,x\,f(x)\,dx$$

$$= \int_0^\infty \lim_{n\to\infty} \chi_{[0,n]}(x)\,\frac{1}{n}\,x\,f(x)\,dx$$

$$= \int_0^\infty 0\,dx = 0.$$

(2.) $f \geq 0$ abs. cont on $[0,1]$, $\alpha \geq 1$. Show f^α is abs. cont.

$\mapsto$ we will use the fact that f abs cont, g Lipschitz $\Rightarrow$ $g\circ f$ abs. cont.)

Step 1: $g(x) = x^\alpha$ is Lipschitz: note that g is defined on cpt interval $[0,1]$, hence g and g' are bdd.

By the MVT, for every x, y, $\exists c$ st. $|g(x)-g(y)| = |g'(c)||x-y|$

However, g' is bdd, hence $|g(x)-g(y)| = |g'(c)||x-y| \leq M|x-y|$, hence g is Lipschitz.

Step 2: $g\circ f$ is abs. cont.: f is abs. cont. on $[0,1]$, hence given $\varepsilon > 0$, $\exists \delta > 0$ st. $\sum(y_k - x_k) < \delta \Rightarrow \sum|f(x_k)-f(y_k)| < \varepsilon$; in particular, $\exists \delta_0 > 0$ st. $\sum|f(x_k)-f(y_k)| < \frac{\varepsilon}{M}$; now:

$$\sum|g\circ f(x_k) - g\circ f(y_k)| \leq \sum M|f(x_k)-f(y_k)| < M\left(\frac{\varepsilon}{M}\right) = \varepsilon,$$

hence $g\circ f$ abs. cont.

Other methods?

(3.) (a) (μ_k) sequence of finite, signed measures. Find positive measure μ s.t. $\mu_k \ll \mu$ $\forall k$.

Let $\mu = \sum_{k=0}^{\infty} \frac{|\mu_k|}{|\mu_k|(X) 2^k}$; positive by definition

- See first that for pairwise disjoint sets (E_n),

$$\mu(\bigcup_n E_n) = \sum_{k=0}^{\infty} \frac{|\mu_k|(\bigcup_n E_n)}{|\mu_k|(X) 2^k} = \sum_{k=0}^{\infty} \frac{\sum_{n=0}^{\infty} |\mu_k|(E_n)}{|\mu_k|(X) 2^k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{|\mu_k|(E_n)}{|\mu_k|(X) 2^k} = \sum_{n=0}^{\infty} \mu(E_n)$$, hence ctbly additive, so a measure

- Furthermore, $\mu = \sum_{k=0}^{\infty} \frac{|\mu_k|}{|\mu_k|(X) 2^k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2$, hence finite

Now: Let $\mu(E) = 0 \Rightarrow 0 = \sum_{k=0}^{\infty} \frac{|\mu_k|(E)}{|\mu_k|(X) 2^k} \Rightarrow |\mu_k|(E) = 0$ for all k (since all posit

But $0 = |\mu_k|(E) = \mu_k^+(E) + \mu_k^-(E)$ by Jordan decomp, hence $\mu_k^+(E) = 0 = \mu_k^-(E)$ since both positive. Therefore, $\mu_k(E) = \mu_k^+(E) - \mu_k^-(E) = 0 - 0 = 0$, i.e. $\mu_k \ll \mu$ $\forall k$.

(b) Construct increasing function with set of discont $= \mathbb{Q}$

Let $\mathbb{Q} = \{q_i\}_{i \in \mathbb{N}}$ be an enumeration of the rationals and let $\sum_{n=1}^{\infty} a_n$ be a convergent series with $a_i \geq 0$ $\forall i$

Then let $f(x) = \sum_{m=0}^{\infty} \chi_{[q_m, \infty)}(x) \, a_m$

→ cont'd

3(b) cont'd:

→ f is increasing: Suppose $x<y$. $\mathbb{Q}$ is dense, hence $\exists\, q_k$ such that $y<q_k<x$, hence:

$$f(y)-f(x)=\sum_{m\geq 0}\chi_{[q_m,\infty)}(y)a_m-\sum_{m\geq 0}\chi_{[q_m,\infty)}(x)a_m\geq a_k>0,$$

→ f is continuous at $\mathbb{R}\setminus\mathbb{Q}$: let $p\notin\mathbb{Q}$.

For $\varepsilon>0$, choose N such that $\sum_{n=N+1}^{\infty}a_n<\varepsilon$ (possible since $\sum a_n$ is a convergent series)

Let $r_1=\max(\{q_1,\dots,q_N\}\cap(-\infty,p])$ and $r_2=\min(\{q_1,\dots,q_N\}\cap[p,\infty))$

Then, for $x_1\in(r_1,p)$ and $x_2\in(p,r_2)$, we have

$\{q_1,\dots,q_N\}\cap(x_1,x_2)=\emptyset$, hence since $r_1<x_1<p<x_2<r_2$,

we have: $\left.\begin{array}{l}\chi_{[q_m,\infty)}(x_1)=\chi_{[q_m,\infty)}(x_2)=1 \text{ for all } q_m<r_1\\ \text{and } \chi_{[q_m,\infty)}(x_1)=\chi_{[q_m,\infty)}=0 \text{ for all } q_m>r_2\end{array}\right\}$ (where $q_m\in\{q_1,\dots,q_N\}$)

Hence: $$f(x_2)-f(x_1)=\sum_{m=0}^{\infty}\chi_{[q_m,\infty)}(x_2)a_m-\sum_{m=0}^{\infty}\chi_{[q_m,\infty)}(x_1)a_m$$
$$\leq\sum_{m=N+1}^{\infty}a_m<\varepsilon$$

$x_1<p$, so $\lim_{x\to p^-}f(x)=\lim_{x\to p^-}\sum_{m=0}^{\infty}\chi_{[q_m,\infty)}(x)q_m\geq f(x_1)$

$x_2>p$, so $\lim_{x\to p^+}f(x)=\lim_{x\to p^+}\sum_{m=0}^{\infty}\chi_{[q_m,\infty)}(x)q_m\leq f(x_2)$

and so:

$\lim_{x\to p^+}f(x)-\lim_{x\to p^-}f(x)\leq f(x_2)-f(x_1)<\varepsilon$, which is true for all ε,

hence letting $\varepsilon\to 0$ we get $\lim_{x\to p^+}f(x)=\lim_{x\to p^-}f(x)$, hence f is cont. at p.

→ f is discontinuous at $\mathbb{Q}$: $p\in\mathbb{Q}$. Then $\exists n$ s.t. $q_n=p$. For all $x\in(p,\infty)=(q_n,\infty)$ we may choose a_k s.t. $q_n=p<q_k<x$, hence:

$$f(x)-f(p)=\sum_{m\geq 0}\chi_{[a_m,\infty)}(x)a_m-\sum_{m\geq 0}\chi_{[a_m,\infty)}(p)a_m\geq a_k$$

So $f(x)\geq a_k+f(p)$, hence $\lim_{x\to p^+}f(x)\geq f(p)+a_k>f(p)$, hence discontinuous at p.

④ $A_r f(x) = \frac{1}{m(B_r(x))}\int_{B_r(x)} f(y)\,dy$

$Hf(x) = \sup_{r\geq 0} A_r|f|(x) = \sup_{r\geq 0}\left\{\frac{1}{m(B_r(x))}\int_{B_r(x)}|f(y)|\,dy\right\}$

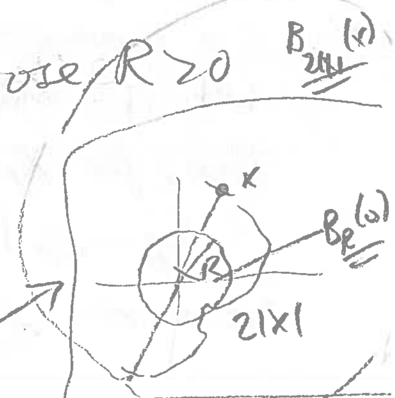
(a) Show for $f\in L^1(\mathbb{R}^n)$, $f\neq 0$, $\exists C, C', R>0$ s.t.

(1) $Hf(x)\geq C|x|^{-n}$ $\forall |x|>R$

(2) $m(\{Hf(x)>\alpha\})\geq \frac{C'}{\alpha}$

Since $f\neq 0$, we have $\int_{\mathbb{R}^n}|f|\neq 0$, hence choose $R\geq 0$ such that

$$\int_{B_{\frac{R}{2}}(0)}|f| > \alpha > 0 \text{ for some } \alpha.$$



→ Now let $|X|>R$, hence

$$Hf(x) = \sup_{r\geq 0}\left\{\frac{1}{m(B_r(x))}\int_{B_r(x)}|f(y)|\,dy\right\} \geq \frac{1}{m(B_{2|x|}(x))}\int_{B_{2|x|}(x)}|f(y)|\,dy$$

$$\geq \frac{1}{m(B_{2|x|}(x))}\int_{B_R(0)}|f(y)|\,dy = \frac{1}{C_1(2|x|)^n}\int_{B_R(0)}|f(y)|\,dy > \frac{\alpha}{C_1 2^n|x|^n}$$

since $B_{2|x|}(x)\supseteq B_R(0)$ for $|x|>R$

Let $C = \frac{\alpha}{C_1 2^n}$

Hence $Hf(x) > \frac{C}{|x|^n}$ (1)

→ Now choose α small enough such that $\frac{C}{\alpha} > R^n$:

Then $\left(\frac{C}{\alpha}\right)^{1/n} > R$, hence may choose x with $R<|x|<\left(\frac{C}{\alpha}\right)^{1/n}$.

Then since $R<|x|$, we have $Hf(x)>\alpha$, hence $\{Hf(x)>\alpha\}\supseteq\{R<|x|<\left(\frac{C}{\alpha}\right)^{1/n}\}$

$\Rightarrow m(\{Hf(x)>\alpha\}) \geq m\left(\{R<|x|<\left(\frac{C}{\alpha}\right)^{1/n}\}\right) = m\left(\{|x|<\left(\frac{C}{\alpha}\right)^{1/n}\}\right) - m(\{|x|\leq R\})$

$= K\left(\frac{C}{\alpha}\right) - KR^n = K\left(\frac{C}{\alpha} - R^n\right)$. Hence let $C' = CK\left(1-\frac{R^n\alpha}{C}\right)$

(and then $\frac{C'}{\alpha} = \frac{CK}{\alpha}\left(1-\frac{R^n\alpha}{C}\right) = K\left(\frac{C}{\alpha}-R^n\right) \leq m(\{Hf(x)>\alpha\})$)

(b.) $H^*f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)|\,dy : B \text{ ball w/ } x\in B\right\}$

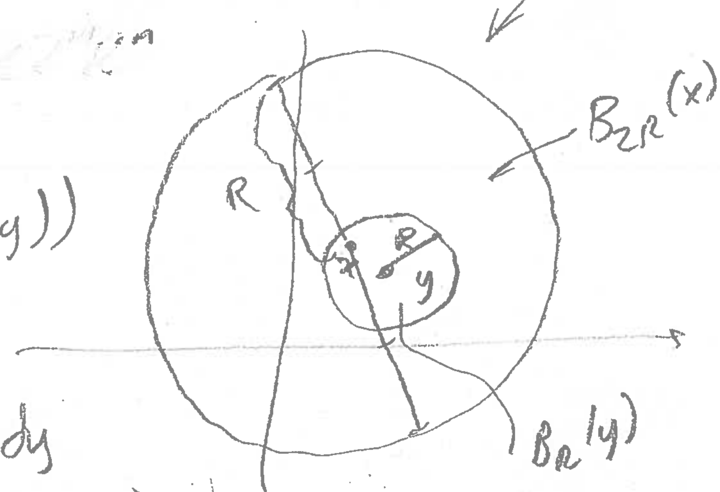
→ $Hf(x) = \sup_{r>0}\left\{\frac{1}{m(B_r(x))}\int_{B_r(x)} |f(y)|\,dy\right\}$

$H^*f(x) = \sup_{B\ni x}\left\{\frac{1}{m(B)}\int_B |f(y)|\,dy\right\}$

Note that Hf takes a sup over a smaller class of balls, hence $Hf(x) \le H^*f(x)$

→ On the other hand, if $x \in B = B_R(y)$, then $B_R(y) \subseteq B_{2R}(x)$ and see that

$$Vol(B_{2R}(x)) = K(2R)^n$$
$$Vol(B_R(y)) = KR^n,$$



hence $Vol(B_{2R}(x)) = 2^n Vol(B_R(y))$

Hence we have:

$$\frac{1}{m(B)}\int_B |f(y)|\,dy = \frac{2^n}{m(B_{2R}(x))}\int_B |f(y)|\,dy$$
$$\le \frac{2^n}{m(B_{2R}(x))}\int_{B_{2R}(x)} |f(y)|\,dy$$

By $B_R(y) \subseteq B_{2R}(x)$

⇒ inequalities preserved after taking sups,

So $H^*f(x) \le 2^n Hf(x)$