Real, Fall 2011

(1) $f \in L^1([0,\infty))$. (Note: we do not need to assume $f \geq 0$).

Find $\lim\limits_{n\to\infty} \frac{1}{n}\int_0^n x f(x)\,dx$.

On $[0,\infty)$ let $f_n = \frac{x}{n}\chi_{[0,n]} f(x)$.

Then $f_n(x) \longrightarrow 0$ everywhere.

Also, $|f_n| = \frac{x}{n}|f(x)|\chi_{[0,n]}$

$= 0$ on $\{x > n\}$ and $\leq |f(x)|$ on $\{0 \leq x \leq n\}$.

Hence $|f(x)| \in L^1([0,\infty))$ is a dominating function for $\{f_n\}$ on $[0,\infty)$. By the DCT

$$\lim_{n\to\infty} \int_0^n x f(x)\,dx = \lim_{n\to\infty} \int_{[0,\infty)} \frac{x}{n}\chi_{[0,n]} f(x)\,dx = 0.$$

$\square$

(2) $f \geq 0$ is absolutely continuous on $[0,1]$ and $\alpha > 1$. Show f^α is absolutely continuous.

We will show y^α is Lipschitz on $[0,1]$.

Assume this is true. Fix $\varepsilon > 0$. Choose $\delta > 0$ s.t. if $\sum_{i=1}^{n} |x_i - y_i| < \delta$, then $\sum_{i=1}^{n} |f(x_i) - f(y_i)| < \varepsilon$.

Then $\left| f(x_i)^\alpha - f(y_i)^\alpha \right| \underset{\text{Lipschitz}}{\leq} C \left| f(x_i) - f(y_i) \right|$ for each i

hence $\sum_{i=1}^{n} \left| f(x_i)^\alpha - f(y_i)^\alpha \right| \leq C \sum_{i=1}^{n} \left| f(x_i) - f(y_i) \right| < C\varepsilon$.

So f^α is absolutely continuous on $[0,1]$.

To see that y^α is Lipschitz, we simply note since $\alpha > 1$ that $\alpha y^{\alpha-1} \leq \alpha$ on $[0,1]$, hence by the MVT $\dfrac{|y^\alpha - x^\alpha|}{|y - x|} \leq \alpha$ for all $x, y \in [0,1]$

Lipschitz results. $\square$

(3) (a) $\{\mu_k\}$ finite signed measures.

Find a positive finite measure μ s.t. $\mu_k \ll \mu$ for all k.

(b) Find an increasing function that is discontinuous exactly on $\mathbb{Q}$.

(a) Set $\mu(E) = \sum_k \frac{|\mu_k|(E)}{|\mu_k|(X)} 2^{-k}$.

This is w.d. since $|\mu_k|(E) \leq |\mu_k|(X) < \infty \ \forall k$.

This is a measure since if $E = \cup E_j$ disjoint,

then $$\sum_k \frac{|\mu_k|(E)}{|\mu_k|(X)} 2^{-k} = \sum_k \sum_l \frac{|\mu_k|(E_l)}{|\mu_k|(X)} 2^{-k}$$

$$= \sum_l \sum_k \frac{|\mu_k|(E_l)}{|\mu_k|(X)} 2^{-k} = \sum_l \mu(E_l),$$

where the swith of sums is justified by positivity (Tonelli). $\square$

(b) Let $\{X_n\}$ be an enumeration of the rationals, and set

$$f(X) = \sum_{X_n \leq X} 2^{-n} \quad (X \in \mathbb{R})$$

which is certainly increasing.

Let $X \in \mathbb{R} - \mathbb{Q}$. We show f is continuous at X. Since f is increasing, hence can only have jump discontinuities, it suffices

choose strictly increasing $X_{n_K} \in \mathbb{Q} \to X$ and show $\lim_{K \to \infty} f(X_{n_K}) = f(X)$.

Indeed, $f(X) = \sum_{X_n \leq X} 2^{-n} = \sup_{\{X_{n_1}, \ldots, X_{n_N}\} \subset \{X_n \leq X\}} \sum_i^N 2^{-n_i}$.

For each $\{X_{n_1}, \ldots, X_{n_N}\}$ there is M s.t. $X_{n_M} \geq \max_i \{X_{n_i}\}$

hence $\sum_i^N 2^{-n_i} \leq \sum_{X_n \leq X_{n_M}} 2^{-n} = f(X_{n_M})$ (since no X_{n_i} equals X)

hence $f(X) \leq \sup_K f(X_{n_K})$, hence $\sup_K f(X_{n_K}) = f(X)$, as desired.

So f is continuous at x. The same argument shows if $x \overset{x_n}{=} \in \mathbb{Q}$, then

$$f(x) = \sup_k f(x_{n_k}) + 2^{-n},$$

hence is not continuous at x. $\square$

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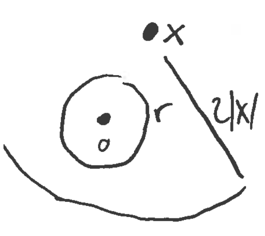
(4) (a)(i) $f \in L^1(\mathbb{R}^n)$, $f \neq 0$.

Show there is $C > 0$, $R > 0$ s.t.

$Hf(x) \geq C|x|^{-n}$ for $|x| > R$.

Since $f \neq 0$, $\exists c > 0$ s.t. $0 < c < \int |f| < \infty$.

Fix $r > 0$. If $|x| > r$, then $B(x, 2|x|) \supset B(0, r)$.



Hence
$$\frac{1}{2^n C_n |x|^n} \int_{B(x, 2|x|)} |f| \geq \frac{1}{2^n C_n |x|^n} \int_{B(0,r)} |f|$$

$$\underset{r \to \infty}{\geq} \frac{c}{2^n C_n |x|^n} \equiv \frac{C}{|x|^n}$$

i.e. $\exists R > 0$ s.t. if $|x| > R$

$$Hf(x) \geq \frac{1}{2^n C_n |x|^n} \int_{B(x, 2|x|)} |f| \geq \frac{C}{|x|^n},$$ as desired. $\square$

(a)(ii) Show there is $C' > 0$ s.t.

$$m\left(\{x : Hf(x) > \alpha\}\right) \geq \frac{C'}{\alpha}.$$

Want to find an annulus on which $Hf(x) > \alpha$. Consider $R < |x| < R'$, where $R' \equiv \left(\frac{C}{\alpha}\right)^{1/n}$ is $> R$ for $\alpha < \frac{C}{R^n}$.

Then $Hf(x) \geq \frac{C}{|x|^n} \geq \frac{C}{(R')^n} = \alpha$. So

$$m\left(\{x : Hf(x) > \alpha\}\right) \geq m\left(\{x : R < |x| < R'\}\right)$$

$$= C_n\left((R')^n - R^n\right) = C_n\left(\frac{C}{\alpha} - R^n\right)$$

$$= \frac{C_n}{\alpha}\left(C - \alpha R^n\right).$$ If we assume in addition that $\alpha < \frac{C}{2R^n}$, then this last term is $\geq \frac{C_n}{\alpha}\left(C - \frac{C}{2}\right)$

$$= \frac{1}{\alpha}\frac{C_n C}{2} \equiv \frac{C'}{\alpha},$$ as desired. $\square$

(b) Let $H^*f(x) = \sup\left\{ \frac{1}{m(B)} \int_B |f(y)|\,dy : \begin{array}{l} B \text{ is a ball} \\ \text{containing } x \end{array} \right\}$.

Show $Hf \le H^*f \le 2^n Hf$.

Since, in particular, $B(x,r)$ is a ball containing x, so $Hf \le H^*f$. The other inequality follows from $x \in B(y,r) \subset B(x,2r)$

as $\begin{cases} |x'-y| < r \\ |x-y| < r \end{cases} \Longrightarrow |x-x'| \le |x-y| + |x'-y| = 2r$.

So $$\frac{1}{C_n r^n} \int_{B(y,r)} |f(y)|\,dy \le \frac{2^n}{C_n (2r)^n} \int_{B(x,2r)} |f(y)|\,dy \le 2^n Hf(x).$$

$\square$