Fall - 2010:

① $\mathcal{A}$ collection of pairwise-disjoint subsets of σ-finite measure space. Suppose every $E\in\mathcal{A}$ has $\mu(E)>0$. Show $\mathcal{A}$ is at most countable.

σ-finite $\Rightarrow X=\bigcup_{i=1}^{\infty}X_i$ where $\mu(X_i)<\infty\ \forall i$.

Then $A=\bigcup_{i=1}^{\infty}(A\cap X_i)$ for each $A\in\mathcal{A}$, but $\mu(A)>0$, hence $\sum_{i=1}^{\infty}\mu(A\cap X_i)\geq\mu(A)>0$, hence $\mu(A\cap X_i)>0$ for some i.

Now suppose that $\mathcal{A}$ is uncountable:

since there are only countably many X_i, this means there must be some X_i s.t. there are uncountably many $A\in\mathcal{A}$ with $A\cap X_i\neq\phi$ and $\mu(A\cap X_i)>0$.

Call this family $\mathcal{C}\subseteq\mathcal{A}$.

The $A\in\mathcal{C}$ are disjoint since $\mathcal{C}\subseteq\mathcal{A}$, hence the $A\cap X_i$ are also disjoint. Now see that:

$$X_i\supseteq\bigcup_{A\in\mathcal{C}}A\cap X_i$$

since disjoint, uncountable sum of positives

$$\Rightarrow\mu(X_i)\geq\mu\Big(\bigcup_{A\in\mathcal{C}}A\cap X_i\Big)=\sum_{A\in\mathcal{C}}\mu(A\cap X_i)=\infty$$

which is a contradiction since $\mu(X_i)<\infty$. So $\mathcal{A}$ is at most countable

② (a) m Leb. on $\mathbb{R}$, f integrable. Show for $a>0$:

$$\int f(ax)\,dm=\frac{1}{a}\int f(x)\,dm$$

Case: simple functions: let $f=\chi_{[b,c]}$ be a characteristic function.

Then: $\int f(ax)\,dm=\int\chi_{[b,c]}(ax)\,dm=\int\chi_{[\frac{b}{a},\frac{c}{a}]}(x)\,dm=\frac{c}{a}-\frac{b}{a}$

$=\frac{1}{a}(c-b)=\frac{1}{a}\int\chi_{[b,c]}(x)\,dm=\frac{1}{a}\int f(x)\,dm.$

Hence also true for simple functions (linear comb. of char. fns)

Recall that f is integrable; thus $\exists$ seq of simple functions ϕ_n converging monotonically to f. If $f\geq0$ so:

Case: f positive: $\int f(ax)\,dm=\int\lim_{n\to\infty}\phi_n(ax)\,dm\overset{MCT}{=}\lim_{n\to\infty}\int\phi_n(ax)\,dm$

$=\lim_{n\to\infty}\frac{1}{a}\int\phi_n(x)\,dm\underset{MCT}{=}\frac{1}{a}\int\lim_{n\to\infty}\phi_n(x)\,dm=\frac{1}{a}\int f(x)\,dm$ ✓

General case: $f=f_+-f_-$

(b.) F meas on $\mathbb{R}$ s.t. $|F(x)| \le C|x|$ $\forall x$. Suppose F diff. at 0.

Show: $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{nF(x)}{x(1+n^2x^2)}\,dm = \pi F'(0)$

Let $f(x) = \frac{nF(x)}{x(1+n^2x^2)}$ and apply part (a):

$$n\int \frac{nF(x)}{x(1+n^2x^2)}\,dm = \int nf(x) = \int f\left(\frac{x}{n}\right) = \int \frac{nF(\frac{x}{n})}{(\frac{x}{n})(1+n^2(\frac{x}{n})^2)}$$

$$= \int \frac{n^2F(\frac{x}{n})}{x(1+x^2)} \Rightarrow \underline{\int \frac{nF(x)}{x(1+n^2x^2)}\,dm = \int \frac{nF(\frac{x}{n})}{x(1+x^2)}\,dm}$$

Now: $\left|\frac{nF(\frac{x}{n})}{x(1+x^2)}\right| \le \frac{nC|\frac{x}{n}|}{|x(1+x^2)|} = \frac{C}{(1+x^2)} \in L^1(m)$

$\to$ so we may apply DCT:

$$\lim_{n\to\infty} \int \frac{nF(\frac{x}{n})}{x(1+x^2)}\,dm = \int \left(\lim_{n\to\infty} \frac{nF(\frac{x}{n})}{x(1+x^2)}\right)dm = \int \left(\frac{1}{(1+x^2)} \lim_{n\to\infty} \frac{F(\frac{x}{n})}{(\frac{x}{n})}\right)dm$$

Now, $|F(0)| \le C|0| = 0$, hence $F(0)=0$, and so:

$$\int \left[\frac{1}{(1+x^2)}\left(\lim_{n\to\infty} \frac{F(\frac{x}{n})-F(0)}{\frac{x}{n}-0}\right)\right]dm = \int \left[\frac{1}{(1+x^2)}F'(0)\right]dm = F'(0)\pi$$

Hence: $\underline{\lim_{n\to\infty} \int \frac{nF(x)}{x(1+n^2x^2)}\,dm = \lim_{n\to\infty} \int \frac{nF(\frac{x}{n})}{x(1+x^2)}\,dm = \pi F'(0)}$

(3.) (X,μ) meas, $\mu(X)<\infty$, f meas, $|f|<1$.

<u>Show</u>: $\lim_{n\to\infty}\int_X (1+f+\cdots+f^n)\,d\mu$ exists

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- <u>case $f\geq 0$</u>: let $g_n=\sum_{i=0}^{n} f^i$. Then $g_n=\frac{1-f^{n+1}}{1-f}$ (since $|f|<1$)

and $\lim_{n\to\infty} g_n=\frac{1}{1-f}$; clearly g_n increases monotonically since $f\geq 0$, so apply MCT.

$$\lim_{n\to\infty}\int_X (1+\cdots+f^n)\,d\mu=\lim_{n\to\infty}\int_X g_n\,d\mu \overset{\text{MCT}}{=} \int_X \lim_{n\to\infty} g_n\,d\mu$$

$= \int_X \frac{1}{1-f}\,d\mu<\infty$ since $\frac{1}{1-f(x)}<\infty$ for all x since $|f|<1$

- <u>general case</u>: let $f=f_+-f_-$ where f_+,f_- nonnegative and such that one is always zero;

Then $(f_+-f_-)^k=f_+^k+(-1)^k f_-^k+\underbrace{\cancel{\text{(cross-terms)}}}_{=0}$ (since either $f_+(x)=0$ or $f_-(x)=0$ $\forall x$)

So we have:

$$\lim_{n\to\infty}\int_X (1+\cdots+f^n)\,d\mu=\lim_{n\to\infty}\int_X (1+\cdots+(f_+-f_-)^n)\,d\mu$$

$$=\lim_{n\to\infty}\int_X (1+\cdots+f_+^n+(-1)^n f_-)\,d\mu$$

$$=\lim_{n\to\infty}\left[\int_X (1+f_+ +\cdots+f_+^n)\,d\mu+\int_X (-f_-+f_-^2+\cdots+(-1)^n f_-^n)\,d\mu\right] \quad (*)$$

Now, note that $-f_-+f_-^2+\cdots+(-1)^n f_-^n\leq \cancel{1}+f_-+f_-^2+\cdots+f_-^n$ since $f_-\geq 0$

hence by comparison:

$$(*)\leq \lim_{n\to\infty}\left[\int_X (1+\cdots+f_+^n)\,d\mu+\int_X (1+\cdots+f_-^n)\,d\mu\right]$$

$$=\lim_{n\to\infty}\int_X (1+\cdots+f_+^n)\,d\mu+\lim_{n\to\infty}\int_X (1+\cdots+f_-^n)\,d\mu$$

Both of which exist by the $f\geq 0$ case.

(4) (F_j) seq of non-negative, non-decreasing, right-continuous functions on $[a,b]$ and suppose $F(x) = \sum_{j=1}^{\infty} F_j(x)$.

Show: $F'(x) = \sum_{j=1}^{\infty} F_j'(x)$ for m-a.e. $x \in [a,b]$.

Since the F_j are non-decreasing, so is F, hence F is differentiable a.e. Furthermore, the F_j are non-decreasing and right continuous hence so is F, hence the measures μ_F and μ_{F_j} are well-defined. Since F, F_j are rt-cont. on a closed interval they are also bdd, hence NBV, hence

$\mu_F \ll m \iff F$ abs cont.

$\mu_{F_j} \ll m \iff F_j$ abs cont.

Now, suppose $m(E) = 0 \Rightarrow 0 = \inf\{\sum(b_j - a_j) : E \subseteq \bigcup(a_j, b_j)\}$

Then: $\mu_F(E) = \inf\{\sum \mu_F((a_j, b_j)) : E \subseteq \bigcup(a_j, b_j)\}$

$= \inf\{\sum F(b_j) - F(a_j) : E \subseteq \bigcup(a_j, b_j)\}$

$= 0$ since the first infemum implies that $a_j = b_j$ for each interval.

$\to$ Similarly for F_j.

Hence $\mu_F \ll m$, $\mu_{F_j} \ll m$, hence $F_j(b) - F_j(a) = \int_a^b F_j'(x)dx$

$F(b) - F(a) = \int_a^b F'(x)dx$,

Thus:

by Tonelli since all positive

$$\int_a^b \sum F_j'(x) = \sum \int_a^b F_j'(x) = \sum F_j(b) - F_j(a)$$
$$= F(b) - F(a)$$

hence $\sum F_j'(x) = F'(x)$