Real, Fall 2010

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(1) Show if m is σ-finite and each set in a collection $\mathcal{A}$ consisting of disjoint sets has positive measure, then $\mathcal{A}$ is countable.

Suppose $\mathcal{A}$ is uncountable.

Assume m is finite. Let

$$S = \left\{ \sum_{A \in \mathcal{A}'} m(A) : \begin{array}{l} \mathcal{A}' \subset \mathcal{A} \text{ is countable} \\ \mathcal{A}' \text{ consists of disjoint sets} \end{array} \right\}.$$

Let $m < \infty$ be $\sup S$. Let $\mathcal{A}'_n$ be s.t.

$$m_n = \sum_{A \in \mathcal{A}'_n} m(A) \longrightarrow m \quad \text{as } n \to \infty.$$

Set $\mathcal{B}' = \bigcup_{m=1}^{\infty} \mathcal{A}'_m$. Since the $\mathcal{A}'_m$ are countable, so is $\mathcal{B}'$, hence

$$\underset{\substack{\vee| \\ m_n}}{\sum_{B \in \mathcal{B}'} m(B)} \leq \sup S = m.$$

But since $m_n \to m$, we have $\sum_{B \in \mathcal{B}'} m(B) = m$. Since $\mathcal{A}$ is uncountable and consists of disjoint sets of positive measure, there exists $B^* \notin \mathcal{B}'$ s.t.

$\sum_{B \in \mathcal{B}' \cup \{B^*\}} \mu(B) > m$, a contradiction. Hence, $\mathcal{A}$ is countable.

We reduce to the finite case. Let $X = \bigcup E_j$ disjoint, $\mu(E_j) < \infty$. (unnecessary: We claim at least one of $\mathcal{A}_j = \{A \cap E_j : A \in \mathcal{A}\}$ is uncountable, if $\mathcal{A}$ is. Otherwise, they are all countable. But $\mathcal{A} = \{\bigcup_j A \cap E_j : A \in \mathcal{A}\}$ is obtained by unioning sets from the $\mathcal{A}_j$, so $\mathcal{A}$ would be countable.) Moreover, we claim at least one of $\mathcal{A}_j' = \{A \cap E_j : A \in \mathcal{A} \text{ and } \mu(A \cap E_j) > 0\}$ is uncountable.

~~Combining w/ the reasoning from before, if all such sets were countable, then $\mathcal{A}$, an uncountable collection of disjoint sets of positive measure, could be reconstructed by taking a countable union of disjoint sets of positive (or zero) measure~~

Each $A \in \mathcal{A}$ is a countable union of sets belonging to the $\mathcal{A}_j$; moreover, this partitions the $\mathcal{A}_j$, so that every $E \in \mathcal{A}_j$ corresponds to exactly one A. If there were only a countable number of sets of positive measure among the $\mathcal{A}_j$ (i.e. $\mathcal{A}_j'$ is countable for all j), then only a countable number of the A can have positive measure, as each $E \in \mathcal{A}_j'$ corresponds to exactly one A and A is a countable union. This is a contradiction. So, there is an E_j s.t. $\mathcal{A}_j$ has an uncountable subset of disjoint positive measure sets; applying the finite case to E_j shows $\mathcal{A}_j'$, and hence $\mathcal{A}_j$, is uncountable.

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(2)(a) $f \in L^1(m)$ Lebesgue integrable on $\mathbb{R}$.

Show if $a > 0$, then

$$\int f(ax)\,dm = \frac{1}{a}\int f(x)\,dm.$$

It suffices to prove the result for characteristic functions: by linearity, this extends to nonnegative simple functions; by approximation from below by simple functions + MCT, this extends to nonnegative functions; by linearity, this extends to real functions; by linearity, this extends to complex functions.

So assume $f = \chi_E$, $m(E)<\infty$. Then

$$f(ax) = \begin{cases} 1 & \text{if } ax \in E \\ 0 & \text{if } ax \notin E \end{cases} = \chi_{\frac{E}{a}}(x).$$

Hence,
$$\int f(ax) = \int \chi_{\frac{E}{a}} = m\left(\frac{E}{a}\right) = \frac{1}{a} m(E)$$
$$= \frac{1}{a}\int \chi_E = \frac{1}{a}\int f. \quad \checkmark$$

(b) If f is a function on $\mathbb{R}$, $|F(x)| \le C|x|$, and $f'(0)$ exists, then

$$\lim_{n\to\infty} \int \frac{nF(x)}{x(1+n^2x^2)}\, dm = \pi F'(0).$$

First, we show $\int \frac{n F(x)}{x(1+n^2x^2)}$ makes sense.

we have $\displaystyle\int \frac{n|F(x)|}{|x|(1+n^2x^2)} \le Cn\int \frac{dx}{1+n^2x^2}$

$$= C \tan^{-1}(nx)\Big|_{-\infty}^{\infty}$$

$$= C\pi < \infty.$$

So $\frac{nf(x)}{x(1+n^2x^2)} \in L^1(m)$.

~~Moreover, $\frac{Cn}{1+n^2x^2}$ is a dominating for $\{g_n\}$, hence by the DCT~~

~~$\lim\limits_{n\to\infty}$~~

By part (a), we may effectively use u-substitution, yielding

$$\int \frac{nF(x)}{x(1+n^2x^2)}\,dx = \int \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)}\,du.$$

$u = nx.$

$dx = \frac{du}{n}.$

Since $\displaystyle\int \frac{|F(\frac{u}{n})|}{|\frac{u}{n}|(1+u^2)} \le C\int \frac{du}{1+u^2} = \pi C,$

we have that $\dfrac{C}{1+u^2}$ is a dominating

function for $\{g_n(u)\}$, $g_n(u) = \dfrac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)}$.

Also, since $|F(x)| \leq C|x|$, we have

$F(0) = 0$, hence

$$F'(0) = \lim_{h \to 0} \frac{F(h)}{h} = \lim_{n \to \infty} \frac{F(\frac{u}{n})}{\frac{u}{n}}.$$

Therefore, by the DCT we conclude

$$\lim_{n \to \infty} \int \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)} = \int \lim_{n \to \infty} \frac{F(\frac{u}{n})}{\frac{u}{n}(1+u^2)}$$

$$= F'(0) \int \frac{du}{1+u^2} = \pi F'(0).$$

□

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(3) $m(X) < \infty$ and $|f| < 1$. Show

$\lim_{n \to \infty} \int 1 + f + \cdots + f^n$ exists. (Possibly ∞.)

~~In this problem we assume f is real-valued.~~

For each n, $\int 1 + f + \cdots + f^n$ makes sense.

Indeed, $$\int \left| 1 + f + \cdots + f^n \right| \leq \int 1 + |f| + \cdots + |f|^n$$
$$\leq (n+1)\mu(X) < \infty.$$

~~To say $\lim_{n\to\infty} \int 1 + f + \cdots + f^n$ exists is to say that the series $\sum_{n=0}^{\infty} \int f^n$ converges.~~

Assume the result is proved for $f \geq 0$ and $f \leq 0$. Then the result follows for f real. Indeed, write $f = f_+ - f_-$. Then $f^n = f_+^n + (-1)^n f_-^n$, so

$$\lim_{n\to\infty} \int 1 + f + \cdots + f^n = \lim_{n\to\infty} \int 1 + f_+ + \cdots + f_+^n \;+$$
$$\lim_{n\to\infty} \int 1 + (-f_-) + \cdots + (-f_-)^n$$

so long as at least one of these limits is finite; we will show the latter limit is always finite. So let us establish the positive and negative cases.

when $f \geq 0$, $1 + f + \cdots + f^n$ is increasing with n, so the MCT shows $\lim_{n\to\infty} \int 1+f+\cdots+f^n$

$= \int \lim_{n\to\infty} 1 + f + \cdots + f^n$ exists (possibly ∞).

when $-1 < f \leq 0$, $1 + f + \cdots + f^n = \dfrac{1-f^{n+1}}{1-f} \leq 1$

since $|f|^{n+1} \leq |f|$. Hence, 1 serves as a dominating function, and we obtain

$$\lim_{n\to\infty} \int 1 + f + \cdots + f^n = \int \lim_{n\to\infty} 1 + f + \cdots + f^n$$

$$= \int \frac{1}{1-f} < \infty$$

(since $\mu(X) < \infty$ and $\frac{1}{1-f} \leq 1$). So we are done. $\square$

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(4) Let $\{F_j\}$ be a sequence of nonnegative nondecreasing right-continuous functions on $[a,b]$ and suppose $F(x) = \sum F_j(x) < \infty$ for all $x \in [a,b]$. Show $F'(x) = \sum F_j'(x)$ for a.e. x.

(and increasing & right continuous)

Since $F \geq 0$ ✓ and $F_j \geq 0$, we may extend F and F_j to functions in NBV by setting $F(x) = \begin{cases} 0, & x < a \\ F(b), & x > b \end{cases}$ and similarly for F_j.

Thus, we may consider the positive Borel measures μ_F and μ_{F_j} satisfying $F(x) = \mu_F((-\infty, x])$ and similarly for F_j. These measures are uniquely determined by this property, hence $\mu_F = \sum \mu_{F_j}$. By Lebesgue-RN let $d\mu_F = d\lambda + f\,dm$ and $d\mu_{F_j} = d\lambda_j + f_j\,dm$.

Now, by the Lebesgue Differentiation Theorem,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\mu_F((x, x+h])}{h} = f$$

a.e., i.e. F is differentiable a.e. and $F'(x) = f(x)$. Similarly $F_j'(x) = f_j(x)$ a.e. Observe

$$\lambda(E) + \int_E f\,dm = \mu(E) = \sum_j \mu_j(E) = \sum \lambda_j(E) + \sum_j \int_E f_j\,dm$$

$$\overset{\text{positivity}}{=} \sum \lambda_j(E) + \int_E \sum_j f_j\,dm.$$

It is easily shown that since λ_j is mutually singular to m, so is $\sum \lambda_j$. Hence, by the uniqueness in the Lebesgue-RN theorem, $\lambda = \sum_j \lambda_j$ and $f = \sum_j f_j$ a.e. That is,

by passing to a set on which F' and F_j' are mutually defined, $F'(x) = \sum_j F'(x)$ a.e. $\square$