## 2010, Fall

**Problem 1.**

Denote by $(X, \mathcal{M}, \mu)$ the measure space, and write $X$ as a countable disjoint union $X = \bigsqcup_{j \in J} X_j$ with $\mu(X_j) < \infty$ for each $j \in J$. Suppose $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ is uncountable. Each $A_\alpha$ has positive measure, so it has $\mu(X_j \cap A_\alpha) > 0$ for some collection of $j$'s in $J$. Since there are uncountable many $\alpha \in I$ but only countably many $j \in J$, by the pigeonhole principle there must be some $j \in J$ and some uncountable subcollection $I' \subset I$ with $\mu(X_j \cap A_\alpha) > 0$ for all $\alpha \in I'$. But then

$$\infty > \mu(X_j) \geq \mu\Big( \bigsqcup_{\alpha \in I'} (X_j \cap A_\alpha) \Big) = \sum_{\alpha \in I'} \underbrace{\mu(X_j \cap A_\alpha)}_{>0},$$

which is impossible since any uncountable sum of positive numbers in infinite. □

**Problem 2.**

(a) Let $a > 0$. Consider a simple function $\varphi = \sum_{j=1}^n a_j \mathbb{1}_{E_j}$, with $\{a_j\}_{j=1}^n \subset \mathbb{R}$ and $\{E_j\}_{j=1}^n \subset \mathcal{B}_\mathbb{R}$ a disjoint collection. Observe that $\mathbb{1}_{E_j}(ax) = \mathbb{1}_{a^{-1}E_j}(x)$ for any $1 \leq j \leq n$, whereby

$$\int \varphi(ax)\mathsf{d}x = \sum_{j=1}^n a_j m(a^{-1}E_j) = \frac{1}{a}\sum_{j=1}^n a_j m(E_j) = \frac{1}{a}\int \varphi(x)\mathsf{d}x.$$

Now suppose $f \in \mathsf{L}^1(\mathbb{R})$ is arbitrary. By decomposing $f = f^+ - f^-$, it's enough to consider the case $f \in \mathsf{L}^+(\mathbb{R})$. Let $\{\varphi_j\}_{j=1}^\infty \subset \mathsf{L}^+(\mathbb{R})$ be a sequence of simple functions with $\varphi_1 \leq \varphi_2 \leq \cdots$ and $\lim_{j\to\infty} \varphi_j = f$. Then

$$\int f(ax)\mathsf{d}x = \lim_{j\to\infty} \int \varphi_j(ax)\mathsf{d}x = \lim_{j\to\infty} \frac{1}{a}\int \varphi_j(x)\mathsf{d}x = \frac{1}{a}\int f(x)\mathsf{d}x$$

by applying monotone convergence twice. □

(b) Set $f(x) := nF(x)/x(1 + n^2x^2)$. Then by (a),

$$\int f(x)\mathsf{d}x = \frac{1}{n}\int f\left(\frac{x}{n}\right)\mathsf{d}x = \frac{1}{n}\int \frac{nF(x/n)}{(x/n)(1 + n^2(x/n)^2)}\mathsf{d}x = \int \frac{1}{1+x^2}\cdot\frac{F(x/n)}{x/n}\mathsf{d}x$$

for any $n \in \mathbb{N}$. Now taking the limit as $n \to \infty$, we may apply dominated convergence since the integrand on the right satisfies

$$\left|\frac{1}{1+x^2}\cdot\frac{F(x/n)}{x/n}\right| \leq \frac{1}{1+x^2}\cdot\frac{nC|x/n|}{|x|} = \frac{C}{1+x^2}$$

and the right-hand side is integrable. Then

$$\lim_{n\to\infty}\int f(x)\mathsf{d}x = \lim_{n\to\infty}\int \frac{1}{1+x^2}\cdot\frac{F(x/n)}{x/n}\mathsf{d}x = \int \frac{1}{1+x^2}\cdot\underbrace{\lim_{n\to\infty}\frac{F(x/n) - F(0)}{(x/n) - 0}}_{=F'(0)}\mathsf{d}x = \pi F'(0),$$

where we used that $F(0) = 0$ since $|F(x)| \leq C|x|$ for all $x \in \mathbb{R}$. □

**Problem 3.**

Assume first that $f \geq 0$. Clearly $1 + f + \cdots + f^n \leq 1 + f + \cdots + f^n + f^{n+1}$ for all $n \in \mathbb{N}$, so by monotone convergence and the geometric series formula,

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n) = \int_X \lim_{n\to\infty} (1 + f + \cdots + f^n) = \int_X \frac{1}{1-f}.$$

The right-hand side always exists since $\mu(X) < \infty$ and $|f| < 1$. Now consider a general measurable function $f = f^+ - f^-$ with $|f| < 1$. We have that $f^j = (f^+ - f^-)^j = (f^+)^j + (-1)^j (f^-)^j$ for any $j \geq 0$ since the product $f^+ f^-$ appearing in the cross terms is always 0. Then

$$\lim_{n\to\infty} \int_X (1 + f + \cdots + f^n) = \lim_{n\to\infty} \int_X [1 + f^+ + \cdots + (f^+)^n] + \lim_{n\to\infty} \int_X [1 - f^- + \cdots + (-1)^n (f^-)^n]$$
$$\leq \lim_{n\to\infty} \int_X [1 + f^+ + \cdots + (f^+)^n] + \lim_{n\to\infty} \int_X [1 + f^- + \cdots + (f^-)^n] = \int_X \frac{1}{1-f^+} + \int_X \frac{1}{1-f^-},$$

and we're done by the nonnegative case since $f^+, f^- \geq 0$. □

**Problem 4.**

For simplicity, denote $F_0 := F$, and let $j \geq 0$. We may write $\mathsf{d}\mu_{F_j} = \mathsf{d}\nu_j + F_j' \mathsf{d}m$, where $m$ denotes the Lebesgue measure and $\nu_j \perp m$, by Lebesgue-Radon-Nikodym. Thus there is some $m$-null $N_j \subset [a,b]$ with $\nu_j([a,b] \setminus N_j) = 0$. Then $N := \bigcup_{j=0}^{\infty} N_j$ is also $m$-null, and for any $E \in \mathcal{B}_{[a,b]}$ disjoint from $N$, we have by monotone convergence that

$$\int_E \sum_{j=1}^{\infty} F_j' \mathsf{d}m = \sum_{j=1}^{\infty} \int_E F_j' \mathsf{d}m = \sum_{j=1}^{\infty} \int_E \mathsf{d}\mu_{F_j} = \sum_{j=1}^{\infty} \mu_{F_j}(E) = \mu_F(E) = \int_E \mathsf{d}\mu_F = \int_E F' \mathsf{d}m.$$

Since $E$ was arbitrary and $N$ is $m$-null, we conclude that $\sum_{j=1}^{\infty} F_j' = F'$ $m$-a.e on $[a,b]$. □