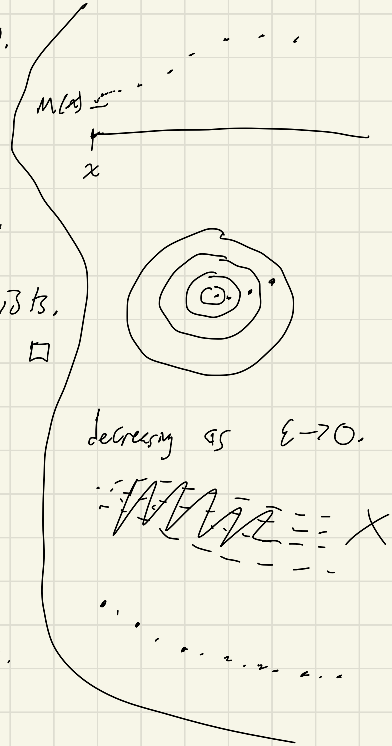
09 Fa. 1

(1) Let f be a bounded function on $\mathbb{R}^n$ and for $\epsilon > 0$ let $M_\epsilon(x) = \sup_{y:|y-x|<\epsilon} f(y)$.
(a) Show that $M(x) = \lim_{\epsilon\to 0} M_\epsilon(x)$ exists for all x.
(b) Show that M is upper semicontinuous, that is, $\limsup_{y\to x} M(y) \le M(x)$.

a) $M_\varepsilon(x) = \sup_{y:|y-x|<\varepsilon} f(y) \ge f(x), \quad \forall \varepsilon > 0.$
and for $\varepsilon_1 > \varepsilon_2 > 0$, we have
$\{y : |y-x| < \varepsilon_1\} \supset \{y : |y-x| < \varepsilon_2\}$
$\Rightarrow M_{\varepsilon_1}(x) \ge M_{\varepsilon_2}(x)$.
That is, $M_\varepsilon(x)$ decreases as $\varepsilon \to 0$.
Thus, since it is also bounded below by $f(x)$ we have $\lim_{\varepsilon\to 0} M_\varepsilon(x)$ exists.

$\square$



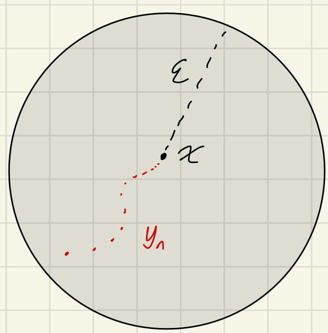
b) Let $y_n \to x$. $\forall \varepsilon > 0$, $\exists N_\varepsilon$ s.t.
$\forall n \ge N_\varepsilon$, $|y_n - x| < \varepsilon$. $\Rightarrow M(y_n) \le M_\varepsilon(x)$
$\forall n \ge N_\varepsilon$, since $\exists$ ball of y_n inside ε-ball of x.
$\limsup_{y_n\to x} M(y_n) = \lim_{N\to\infty} \sup_{y_n : n\ge N} M(y_n) =: L$, say.

For a given N_ε, $L \le \sup_{y_n : n \ge N_\varepsilon} M(y_n) \le M_\varepsilon(x)$.

This holds for any ε.

Thus, $L \le \lim_{\varepsilon\to 0} M_\varepsilon(x) = M(x)$.

$\square$



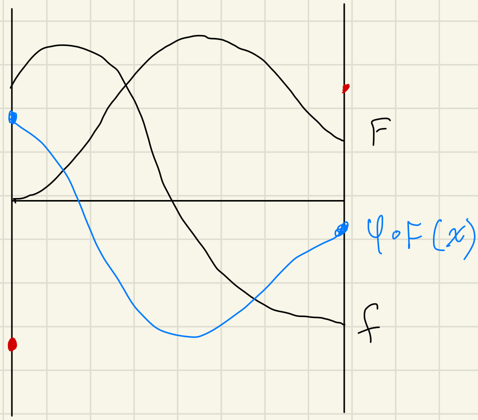
09 Fa. 2

(2) Let m be Lebesgue measure on $[0,1]$, suppose $f \in L^1(m)$ and let $F(x) = \int_0^x f(t)\,dt$. Suppose φ is a Lipshitz function, that is, for some M, $|\varphi(x) - \varphi(y)| \le M|x-y|$ for all x, y. Show that there exists $g \in L^1(m)$ such that $\varphi(F(x)) = \int_0^x g(t)\,dt$.

Maybe $g = \varphi(f(t))$

Does $\varphi(F(x)) = \int_0^x \varphi(f(t))\,dt$?

$$\varphi(F(x)) = \varphi\left(\int_0^x f(t)\,dt\right)$$



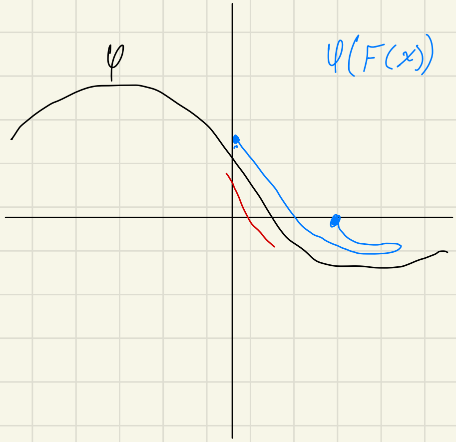
at 0, $\varphi(F(0)) = \varphi(0)$

$\int_0^0 \varphi(f(t))\,dt = 0$ but $\varphi(0) \neq 0$ nec.

$$\frac{d}{dx}\varphi(F(x)) = F'(x)\,\varphi'(F(x)) = f(x)\,\underbrace{\varphi'(F(x))}_{\le M}$$

$$\frac{d}{dx}\int_0^x g(t)\,dt = g(x)$$

$$g(x) = f(x)\,\varphi'(F(x)).$$



Use Lipshitz twice.

FTC (b)

$\exists f$ s.t. $F(x) - F(a) = \int_a^x f(t)\,dt$ ☆

09Fa.3

(3) Let χ_E denote the indicator function of a set E. Suppose $E \subset \mathbb{R}$ has finite Lebesgue measure and define

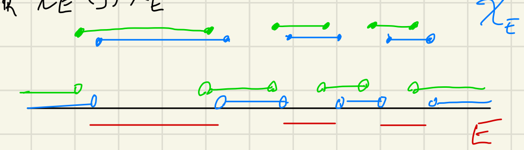
$$f(x) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy.$$

Show that f is continuous.

Let $x_n \to x$.

Claim: $f(x_n) \to f(x)$ or $|f(x_n) - f(x)| \to 0$.

$\rightsquigarrow \lim_{n\to\infty} \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x_n) = \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)$ → just need to show lim taken in.



$$\chi_E(y-x) = \chi_{E+x}(y).$$

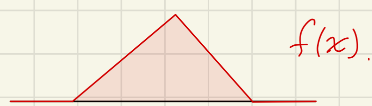
$$\chi_E(y)\,\chi_{E+x}(y) = \chi_{E\cap(E+x)}(y)$$

$$\Rightarrow \int_{\mathbb{R}} \chi_E(y)\chi_E(y-x)\,dy = \int_{\mathbb{R}} \chi_{E\cap(E+x)}(y)\,dy = \mu(E\cap(E+x)).$$

True for interval (a,b):

$$f(x) = \mu((a,b)\cap(a+x,\, b+x))$$
$$= b-(a+x) = (b-a)-x \quad \text{OR } 0,$$
$$= \max\{(b-a)-x,\, 0\}.$$

which is continuous.



E is just a countable union of open intervals?

09Fa.4

(4) Let m be Lebesgue measure on $\mathbb{R}$ and let $f_n, f \in L^1(m)$. Suppose there is a constant C such that $||f_n - f||_1 \le \frac{C}{n^2}$ for all $n \ge 1$. Show that $f_n \to f$ a.e. HINT: Consider the sets

$$\{x : |f_n(x) - f(x)| > \epsilon \text{ for some } n \ge N\}.$$

Def $E(N,\varepsilon) = \{x : |f_n(x) - f(x)| \ge \varepsilon$ for some $n \ge N\}$.

Def $F = \{x : f_n(x) \not\to f(x)\}$

$= \{x :$ for some ε, $\forall N$, $\exists$ $n \ge N$ s.t. $|f_n(x) - f(x)| \ge \varepsilon\}$

$= \bigcup_\varepsilon \{x ; \forall N, |f_n(x) - f(x)| \ge \varepsilon$ for some $n \ge N\}$

For a given $\varepsilon > 0$, $m(\underbrace{\{x ; \forall N, |f_n(x) - f(x)| \ge \varepsilon \text{ for some } n \ge N\}}_{*}) = 0$.

For, consider when this set is not null. Say $m(*) > 0$.

Then $\forall N$, $\exists n \ge N$ s.t. $\int |f_n(x) - f(x)|\, dx \ge \int_* |f_n(x) - f(x)|\, dx$

$\ge \int_* \varepsilon \, dx$

$= \varepsilon m(*)$

But For n large enough, $C/n^2 < \varepsilon m(*)$.

$\limsup_n$

Q: Why n^2?

$\sum_{n=K}^{\infty} \underbrace{m(|f_n - f| > \varepsilon)}_{\le \frac{C}{n^2}}$

Borel-Catelli